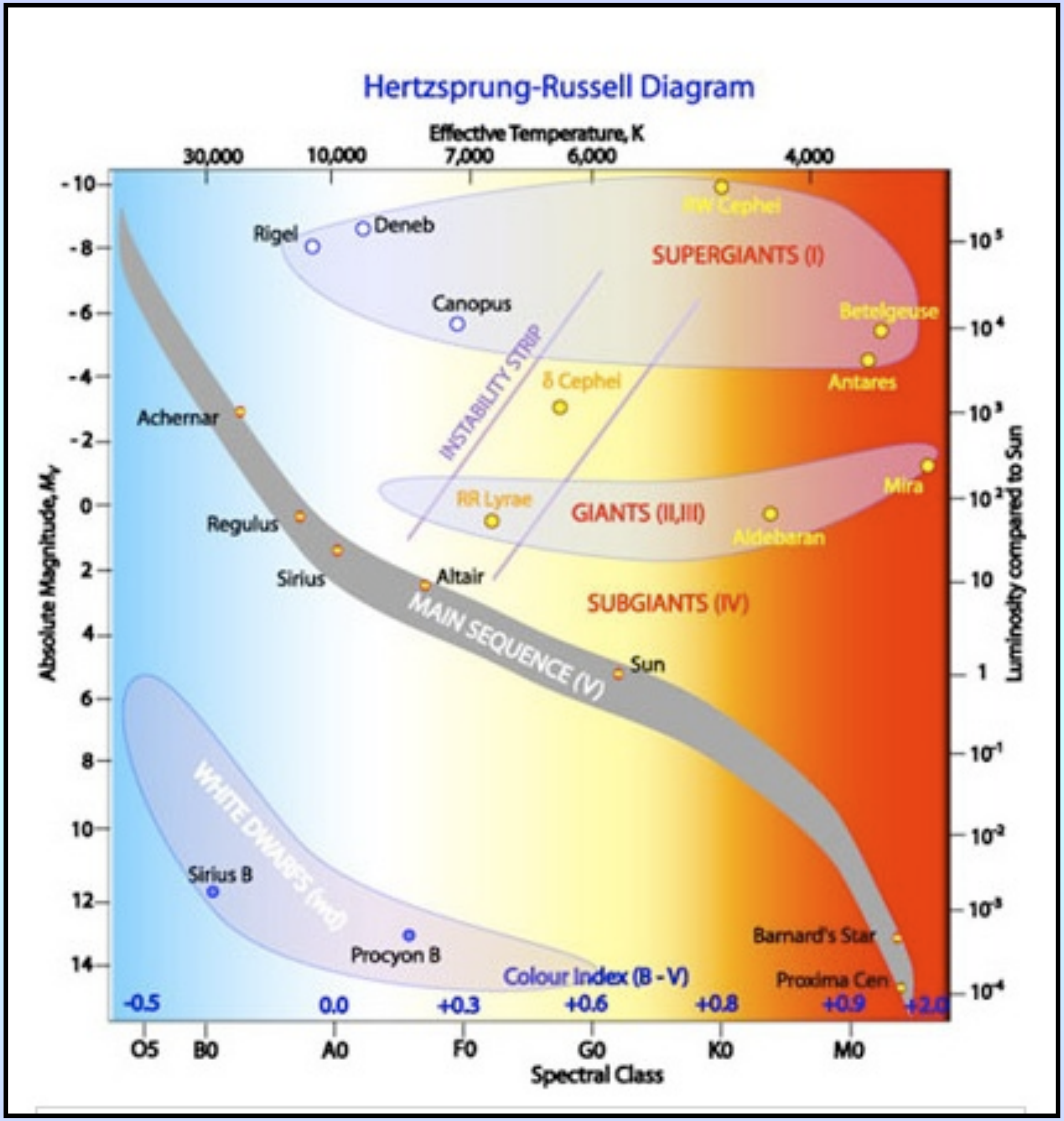
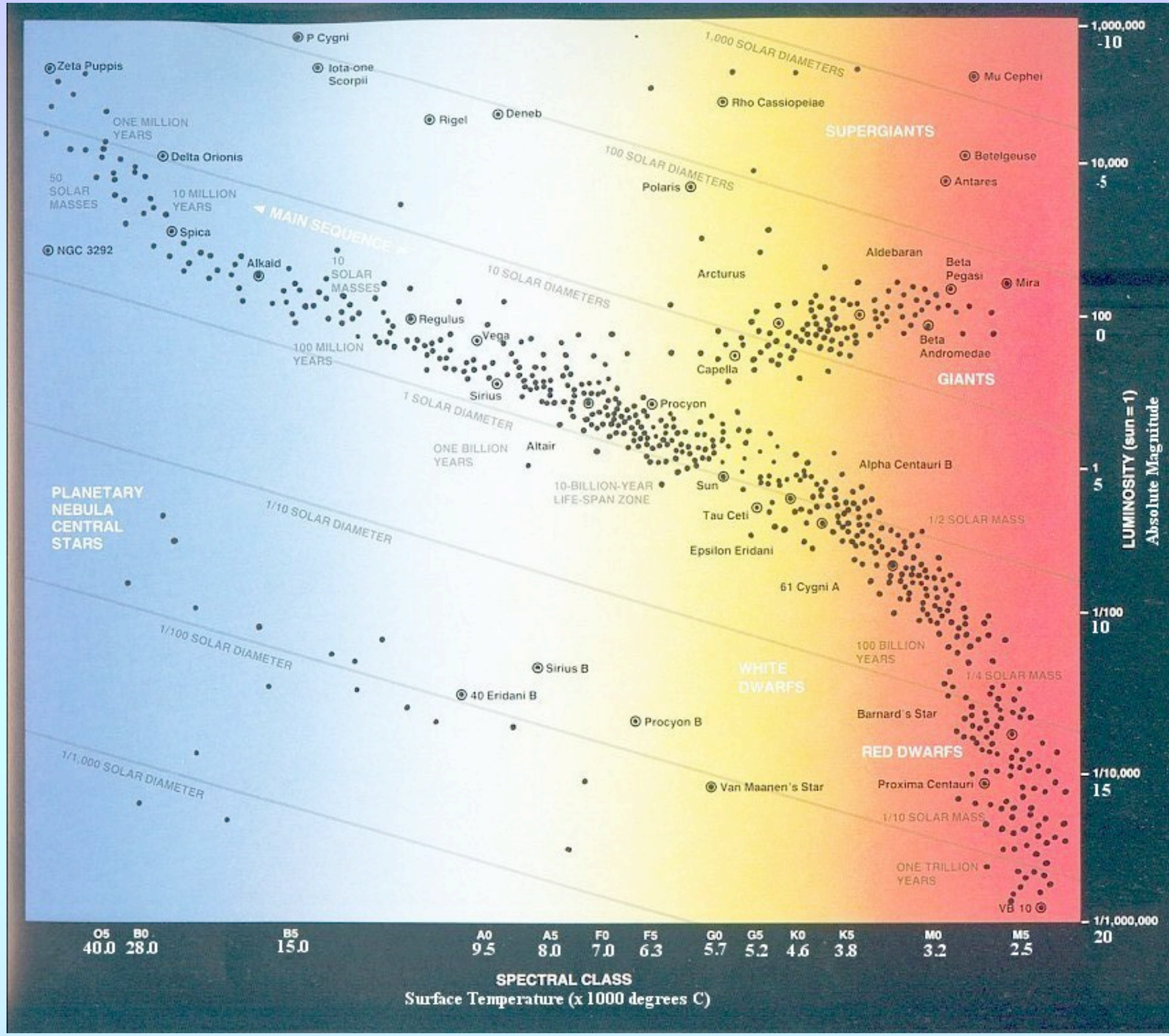
# The HR Diagram

Most (>90%) stars lie on the "main sequence". A few stars are cool and extremely bright, so, by $L = 4 \pi R^2 \sigma T^4$, they must be extremely large. A few stars are hot, but extremely faint, so they must be very small.



# The HR Diagram

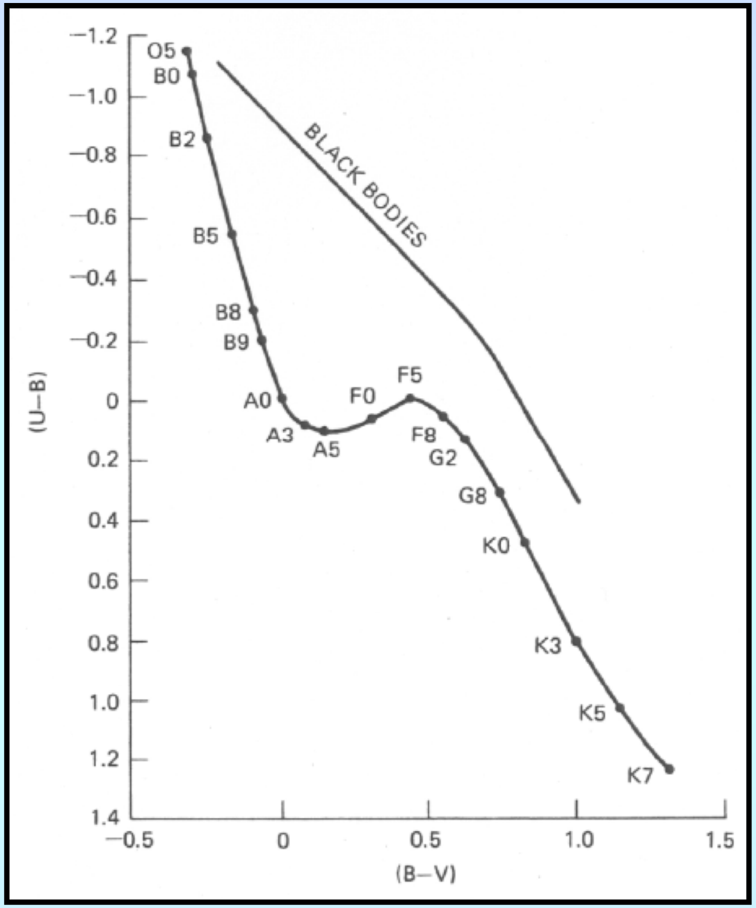
Most (>90%) stars lie on the "main sequence". (But most stars with names you have heard of are giants.) You can see giants much further away!



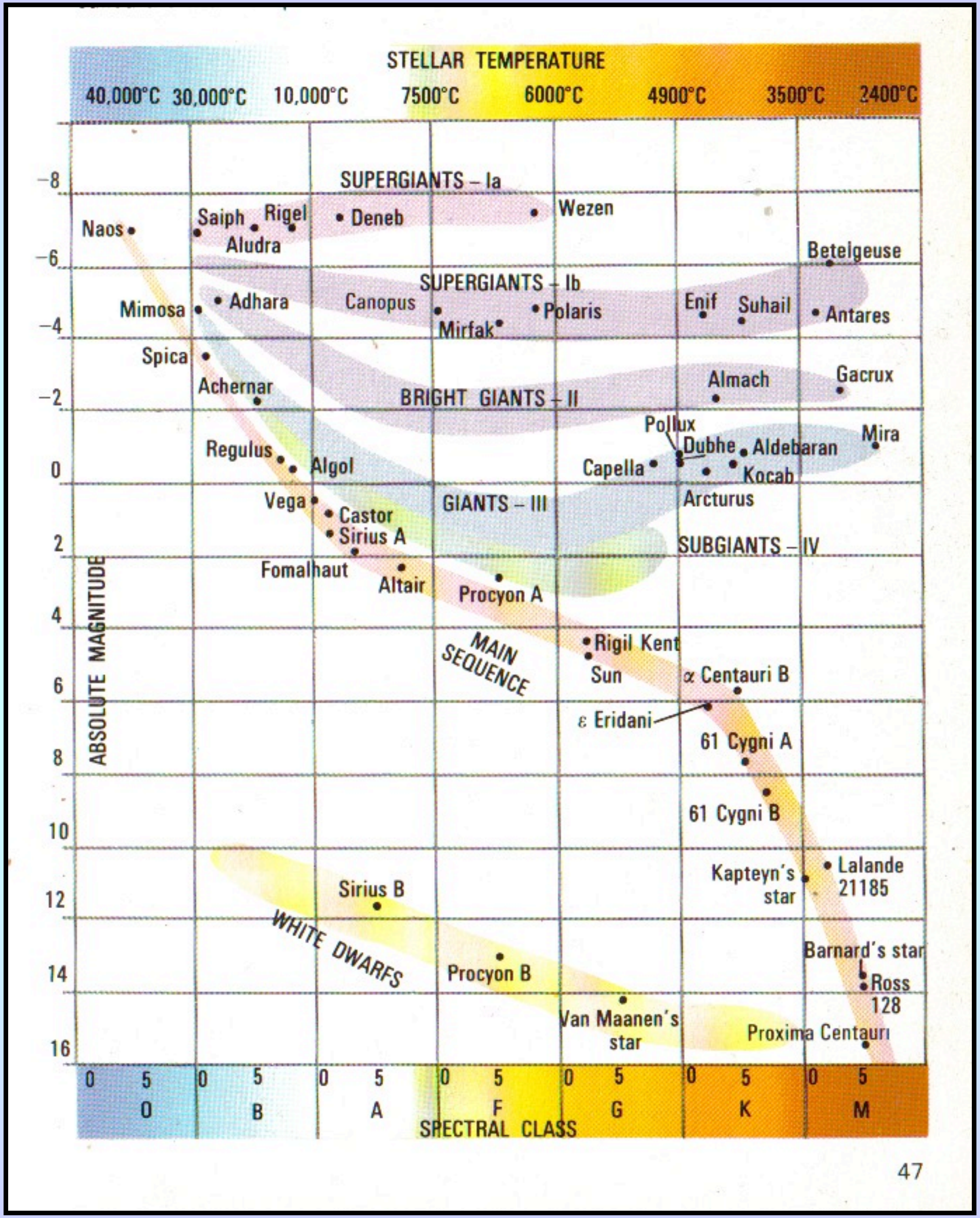

$$f \propto \frac{L}{d^2} \quad \Rightarrow \quad L \propto d^2$$

$$N_{\text{obj}} \propto V \propto d^3 \quad \Rightarrow \quad N_{\text{obj}} \propto L^{3/2}$$

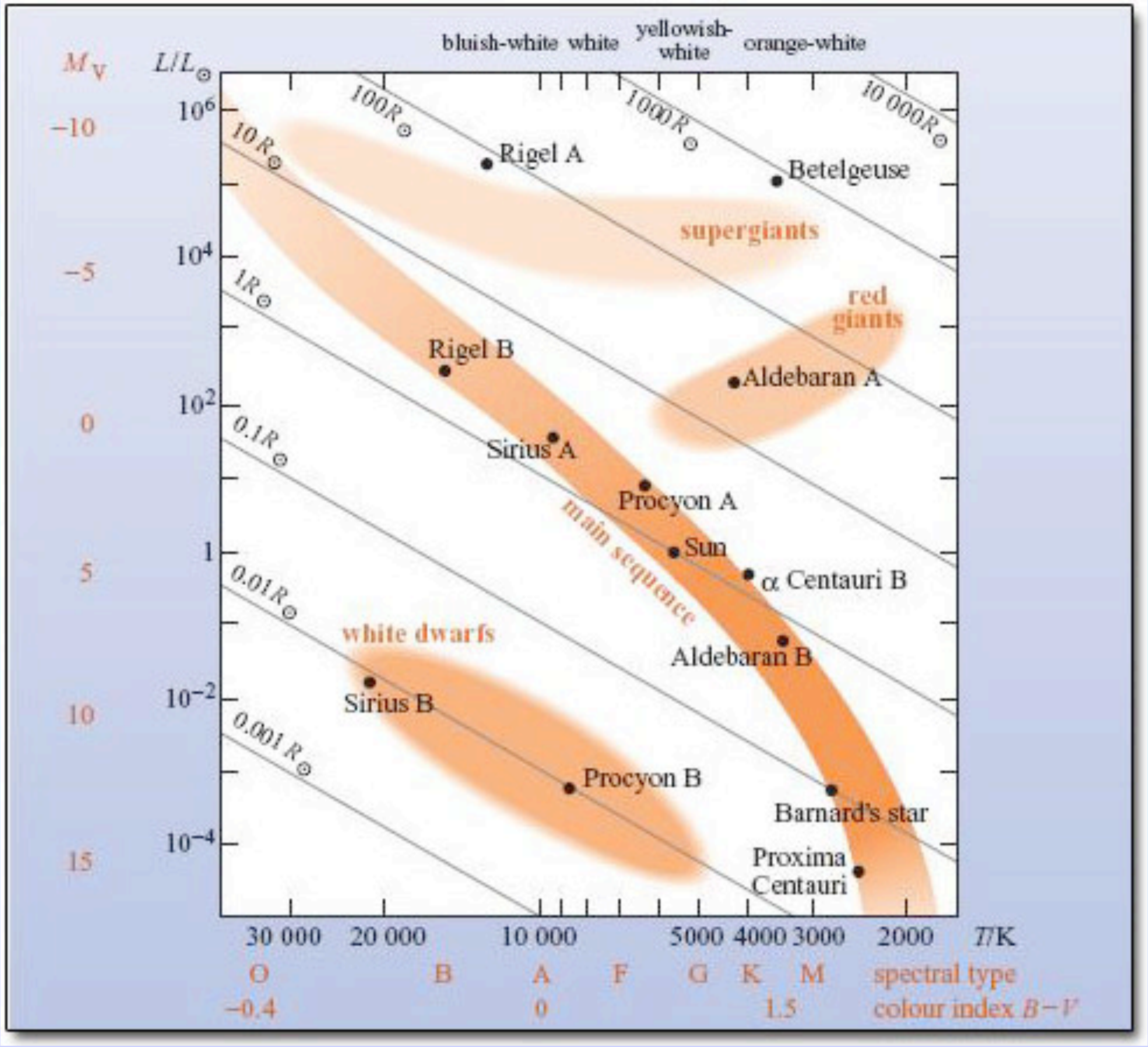
# The HR Diagram




The main sequence in the UBV 2-color diagram. The kink is due to the effects of the Balmer jump.



# The HR Diagram with Iso-Radius Lines

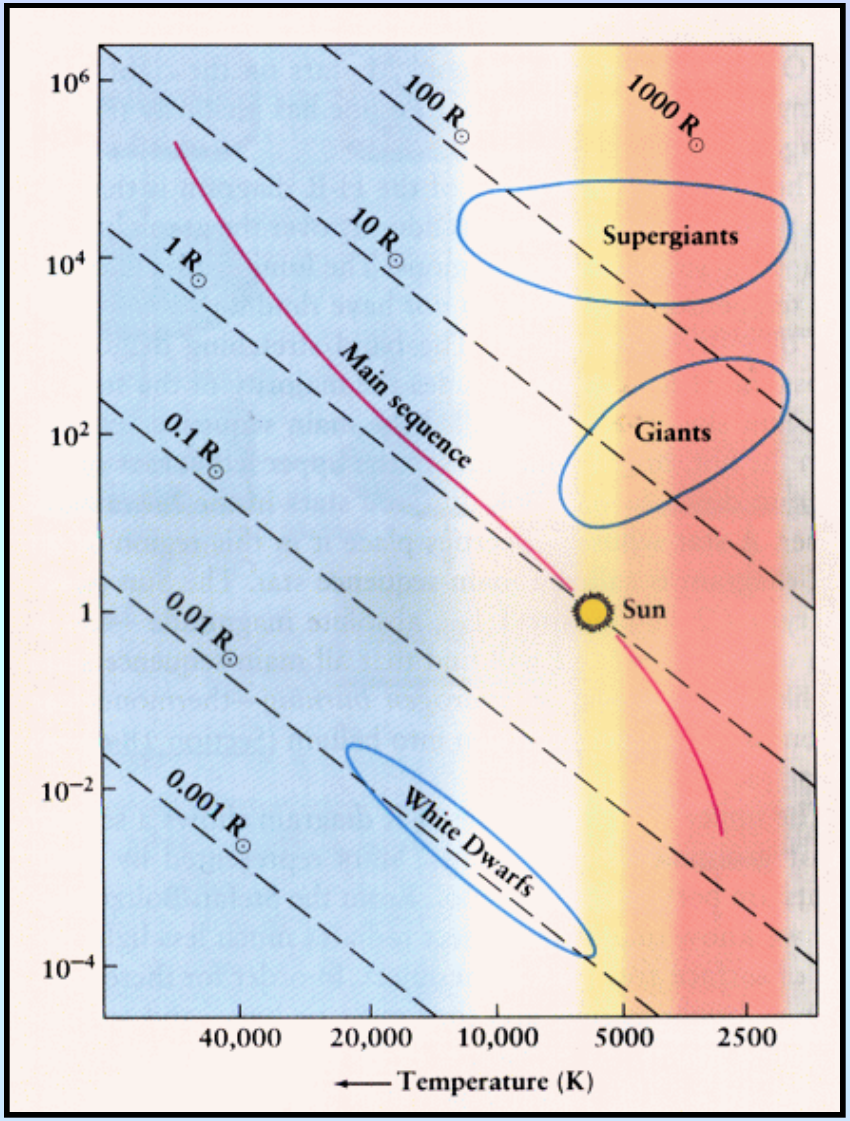


# For Reference

$1\ R_\odot = 6.96 \times 10^{10}$ cm

$1\ R_\oplus = 6.37 \times 10^{8}$ cm $= 0.009\ R_\odot$

1 A.U. $= 1.49 \times 10^{13}$ cm $= 215\ R_\odot$



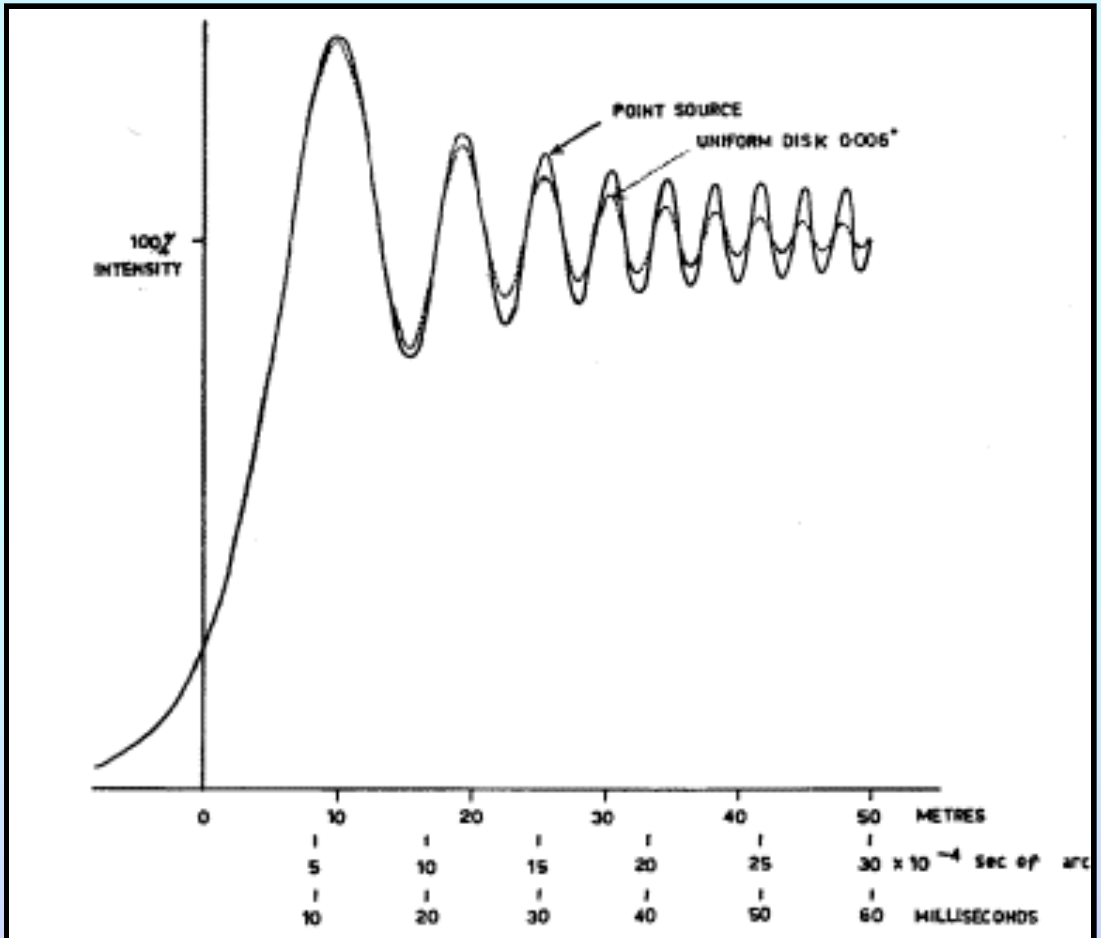
# Determining Stellar Radii

Stellar radii can be estimated by combining measurements of absolute luminosity with temperature. But the uncertainties in each parameter will propagate! Since virtually all stars are unresolvable even at HST (and JWST resolution), there are only a few methods that can directly measure a star's radius:

- Lunar occultations (for angular diameters)
  - Objects must be bright and within 5° of the ecliptic plane
- Interferometry (for angular diameters)
  - Objects must be bright
  - Better in the IR, where the atmosphere is better behaved
- Baade-Wesselink (for pulsating stars)
- Eclipsing binary stars

# Lunar Occultations

The Moon orbits the Earth in ~ 29.5 days, i.e., at a rate of ~ 0.5″/sec. At ~ 10 pc, the angular diameter of the Sun is ~ 0.001 arcsec, which translates into an occultation timescale of a few millisec.




Stars must be bright and within 5° of the ecliptic plane.

Table 7.3 *Some angular diameters measured from lunar occultations according to Ridgeway, Wells and Joyce (1977)*

| HR No.[2] | Star | Sp. type[3] | B − V | $m_V$ | $\Theta_{LD}$[1] $10^{-3}$ arcsec | $\pi$ arcsec | $R/R_\odot$ | $T_{eff}$ |
|---|---|---|---|---|---|---|---|---|
| 2286 | $\mu$ Gem | M3III | 1.64 | 2.88 | 13.65 | 0.020 | 73.2 | 3650 |
| 867 | RZ Ari | M6III | 1.47 | 5.91 | 10.18 | 0.014 | 78.0 | 3160 |
| 1977 | Y Tau | C5II | 3.03 | 6.95 | 8.58 | ? | ? | ? |
| 5301 | | M2III | 1.72 | 4.91 | 4.41 | 0.012 | 39.4 | 4040[4] |
| 4902 | $\psi$ Vir | M3III | 1.60 | 4.79 | 5.86 | 0.021 | 29.9 | 3530 |
| 7150 | $\xi^2$ Sgr | K1III | 1.18 | 3.51 | 3.80 | 0.011 | 37.1 | 4210 |
| 9004 | TX Psc | C5II | 2.60 | 5.04 | 9.31 | −0.004 | ? | ? |
| 3980 | 31 Leo | K4III | 1.45 | 4.37 | 3.55 | ? | ? | 4000 |
| 7900 | $\upsilon$ Cap | M2III | 1.66 | 5.10 | 4.72 | 0.019 | 26.6 | 3440 |

1. Considering the limb darkening of the disk.
2. Number in the Bright Star catalogue, Hoffleit and Jaschek (1982).
3. From the Bright Star catalogue
4. Assuming BC = 2.0.

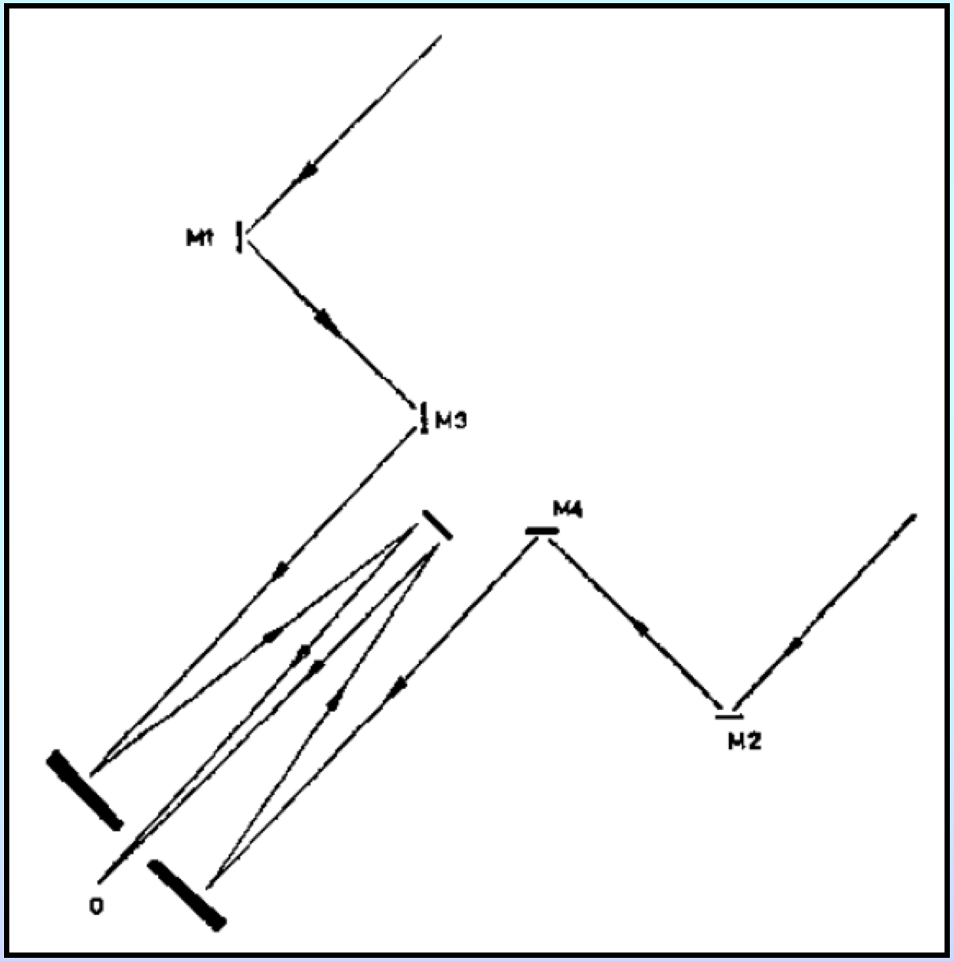
# Interferometry

There are several types of interferometry:

- Amplitude (Michelson) interferometry

Combines the light from two slits (or telescopes) whose separation can be adjusted; object size can be derived by observing the fringes as a function of image separation.

The technique requires matching the path-lengths to a fraction of a wavelength; since atmospheric turbulence destroys image coherence, this technique has not produced many results.



# Interferometry

There are several types of interferometry:

- Amplitude (Michelson)
- Intensity (Hanbury Brown)

At any instant, light from the source produces an interference pattern, which constantly changes (on femto-second timescales). The spatial scale for the interference goes as ~ $\lambda/\theta$, so, at any instant, two detectors separated by less than $\lambda/\theta$ will have their "noise" correlated, while detectors separated by distances greater than $\lambda/\theta$ will have uncorrelated noise. Thus, $\theta$ can be estimated without phase information.

Table 7.2 *Angular diameters, θ, measured with the Hanbury Brown interferometer*

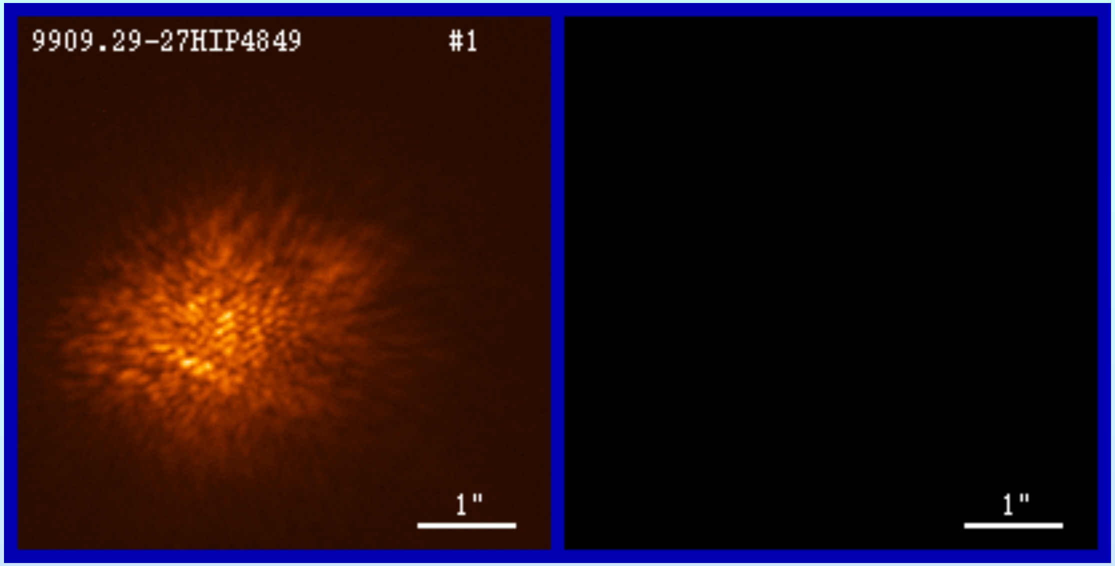
| Star[1] number | Star name | Type[2] | Angular diameter[3] in $10^{-3}$s of arc $\Theta_{LD} \pm \sigma$ | Temperature[4] $[T_{eff} \pm \sigma]$/K |
|---|---|---|---|---|
| 472 | α Eri | B3(Vp) | 1.92 ± 0.07 | 13 700 ± 600 |
| 1713 | β Ori | B8(Ia) | 2.55 ± 0.05 | 11 500 ± 700 |
| 1790 | γ Ori | B2 (III) | 0.72 ± 0.04 | 20 800 ± 1300 |
| 1903 | ε Ori | B0(Ia) | 0.69 ± 0.04 | 24 500 ± 2000 |
| 1948 | ζ Ori | O9.5(Ib) | 0.48 ± 0.04 | 26 100 ± 2200 |
| 2004 | κ Ori | B0.5 (Ia) | 0.45 ± 0.03 | 30 400 ± 2000 |
| 2294 | β CMa | B1 (II-III) | 0.52 ± 0.03 | 25 300 ± 1500 |
| 2326 | α Car | F0 (Ib-II) | 6.6 ± 0.8 | 7500 ± 250 |
| 2421 | γ Gem | A0(IV) | 1.39 ± 0.09 | 9600 ± 500 |
| 2491 | α CMa | A1(V) | 5.89 ± 0.16 | 10 250 ± 150 |
| 2618 | ε CMa | B2 (II) | 0.80 ± 0.05 | 20 800 ± 1300 |
| 2693 | δ CMa | F8 (Ia) | 3.60 ± 0.50 | ... |
| 2827 | η CMa | B5 (Ia) | 0.75 ± 0.06 | 14 200 ± 1300 |
| 2943 | α CMi | F5 (IV-V) | 5.50 ± 0.17 | 6500 ± 200 |
| 3165 | ζ Pup | O5(f) | 0.42 ± 0.03 | 30 700 ± 2500 |
| 3207 | $\gamma^2$ Vel | WC8 + O9 | 0.44 ± 0.05 | 29 000 ± 3000 |
| 3685 | β Car | A1(IV) | 1.59 ± 0.07 | 9500 ± 350 |
| 3982 | α Leo | B7(V) | 1.37 ± 0.06 | 12 700 ± 800 |
| 4534 | β Leo | A3(V) | 1.33 ± 0.10 | 9050 ± 450 |
| 4662 | γ Crv | B8(III) | 0.75 ± 0.06 | 13 100 ± 1200 |
| 4853 | β Cru | B0.5(III) | 0.722 ± 0.023 | 27 900 ± 1200 |
| 5056 | α Vir | B1(IV) | 0.87 ± 0.04 | 22 400 ± 1000 |
| 5132 | ε Cen | B1(III) | 0.48 ± 0.03 | 26 000 ± 1800 |
| 5953 | δ Sco | B0.5(IV) | 0.46 ± 0.04 | ... |
| 6175 | ζ Oph | O9.5(V) | 0.51 ± 0.05 | ... |
| 6556 | α Oph | A5(III) | 1.63 ± 0.13 | 8150 ± 400 |
| 6879 | ε Sgr | A0(V) | 1.44 ± 0.06 | 9650 ± 400 |
| 7001 | α Lyr | A0(V) | 3.24 ± 0.07 | 9250 ± 350 |
| 7557 | α Aql | A7 (IV, V) | 2.98 ± 0.14 | 8250 ± 250 |
| 7790 | α Pav | B2.5(V) | 0.80 ± 0.05 | 17 100 ± 1400 |
| 8425 | α Gru | B7(IV) | 1.02 ± 0.07 | 14 800 ± 1200 |
| 8728 | α PsA | A3(V) | 2.10 ± 0.14 | 9200 ± 500 |

1. Bright star catalog number (Hoffleit and Jaschek, 1982).
2. Spectral type and luminosity class (in brackets), to be discussed in Chapter 10.
3. True angular diameter allowing for the effects of limb-darkening.
4. Effective temperatures will be discussed in Chapter 8.

# Interferometry

There are several types of interferometry:

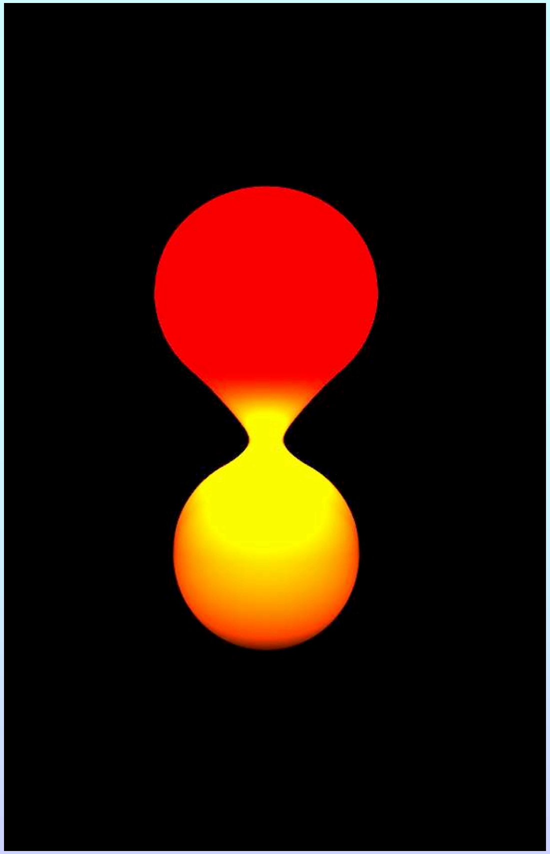
- Amplitude (Michelson)
- Intensity (Hanbury Brown)
- Speckle

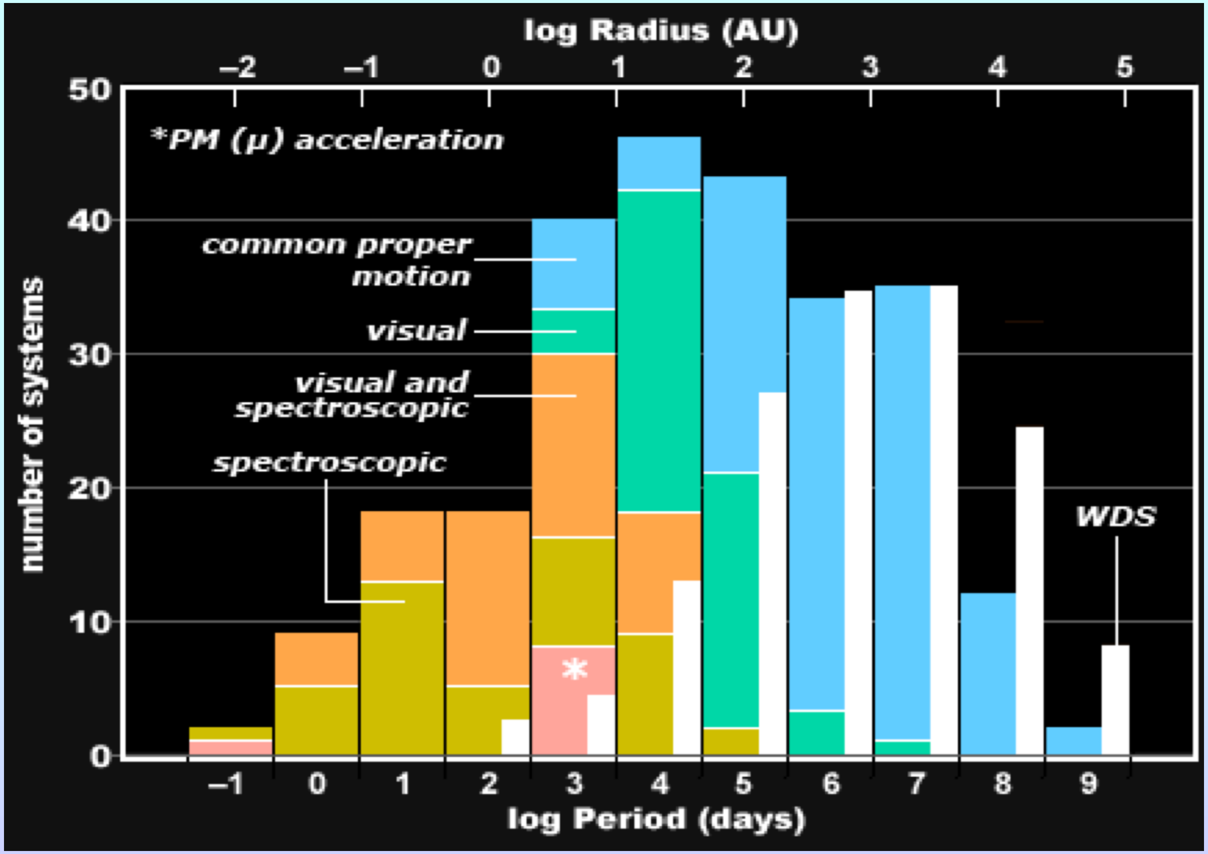



The "seeing" produced by the atmosphere is the superposition of many diffraction-limited images. (In other words, much of the blurring is due to rapid image motion.) By taking a series of very short exposures, and then correlating the images, one can approach the diffraction limit of the telescope. (Adaptive optics also does this.)

# General Information about Binary Stars

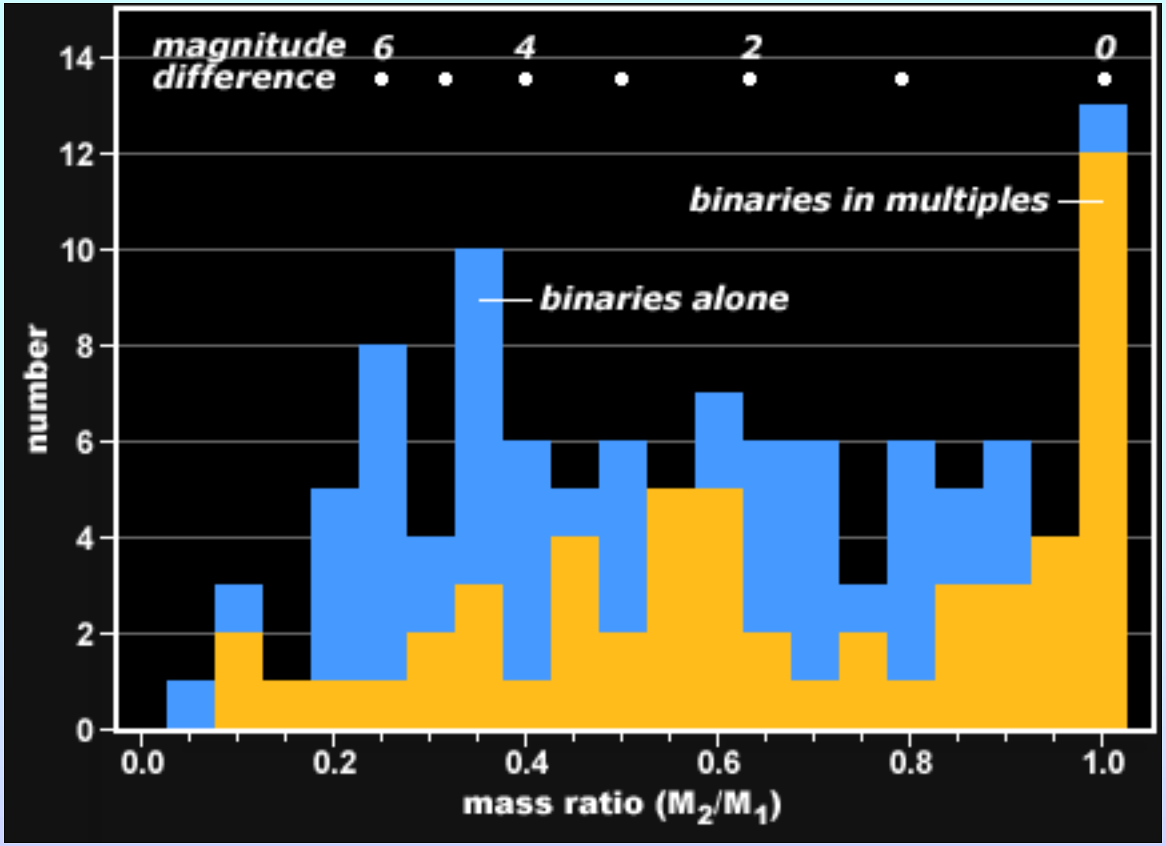
- Perhaps ~ 85% of all stars in the Milky Way are part of multiple systems (binaries, triplets or more)
- The periods (separations) of these binaries span the entire range of possibilities, from contact binaries to separations of ~ 0.1 pc. The distribution of separations appears *roughly* log-normal.





# General Information about Binary Stars

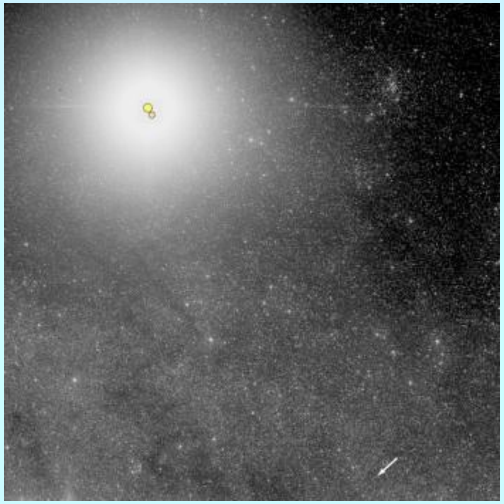
- Perhaps ~ 85% of all stars in the Milky Way are part of multiple systems (binaries, triplets or more)
- The periods (separations) of these binaries span the entire range of possibilities, from contact binaries to separations of ~ 0.1 pc. The distribution of separations appears *roughly* log-normal.
- The possible mass ratios of binary stars span the entire range of masses. There is a possible preference for mass ratios of near 1, but selection effects are important!

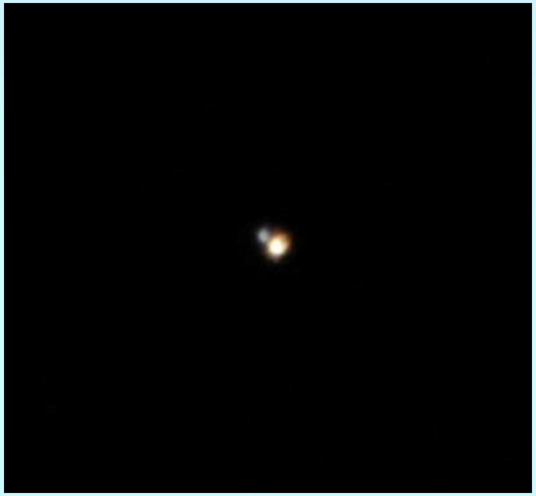


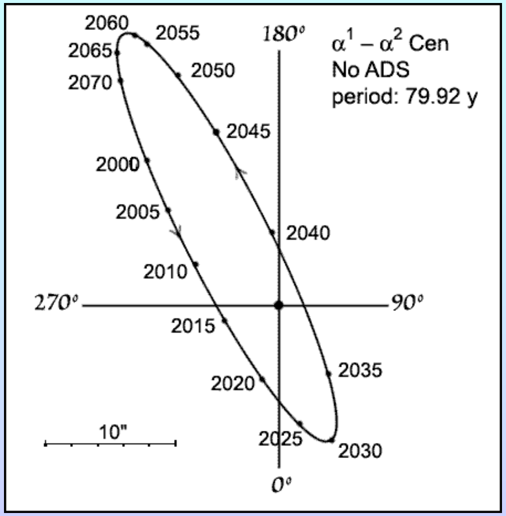
# Binary Stars

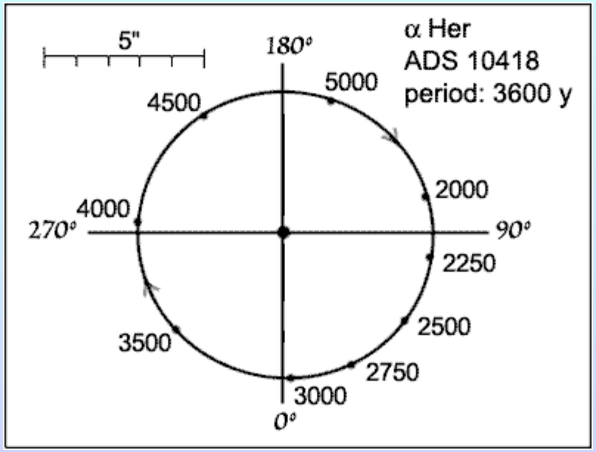
Most direct measurements of stellar radii and stellar masses come from the analysis of binary stars. There are several types of binaries:

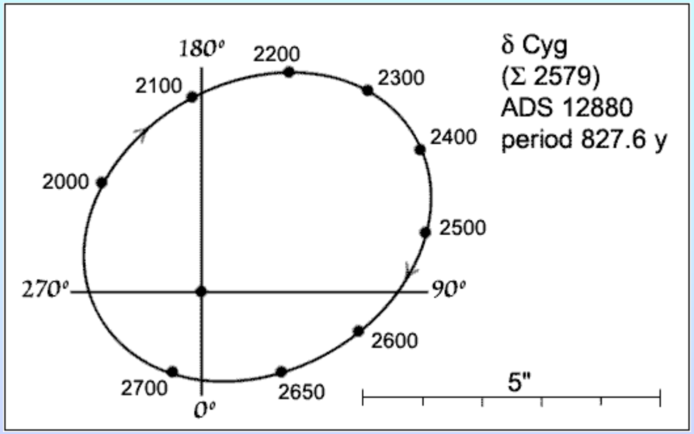
- Visual







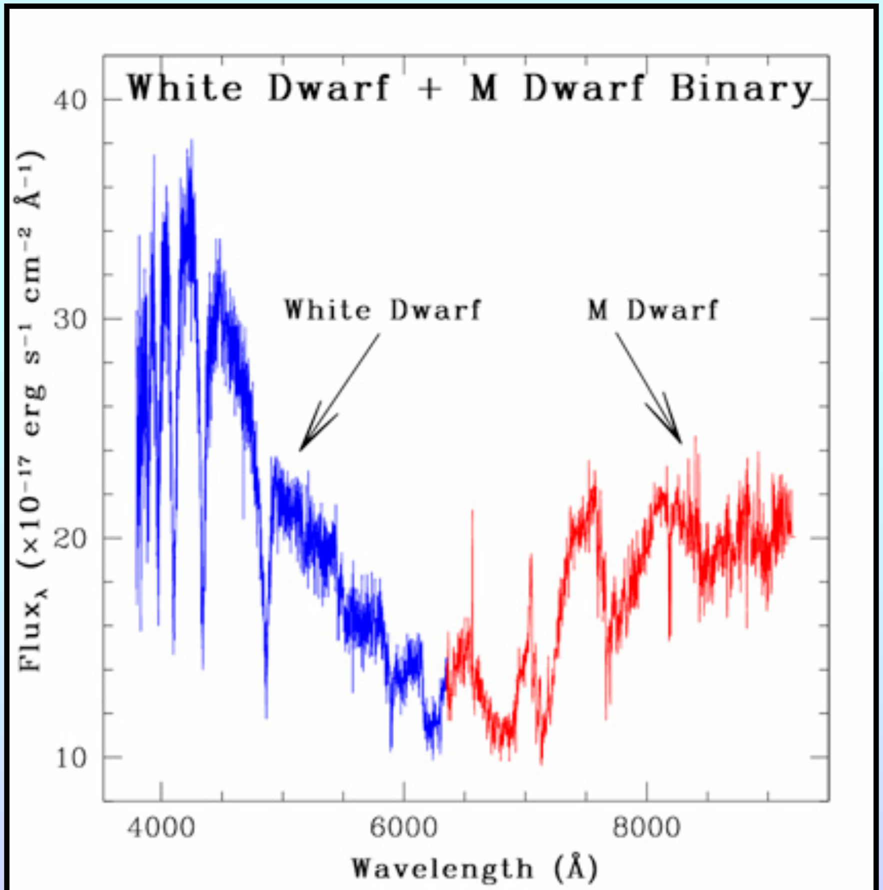



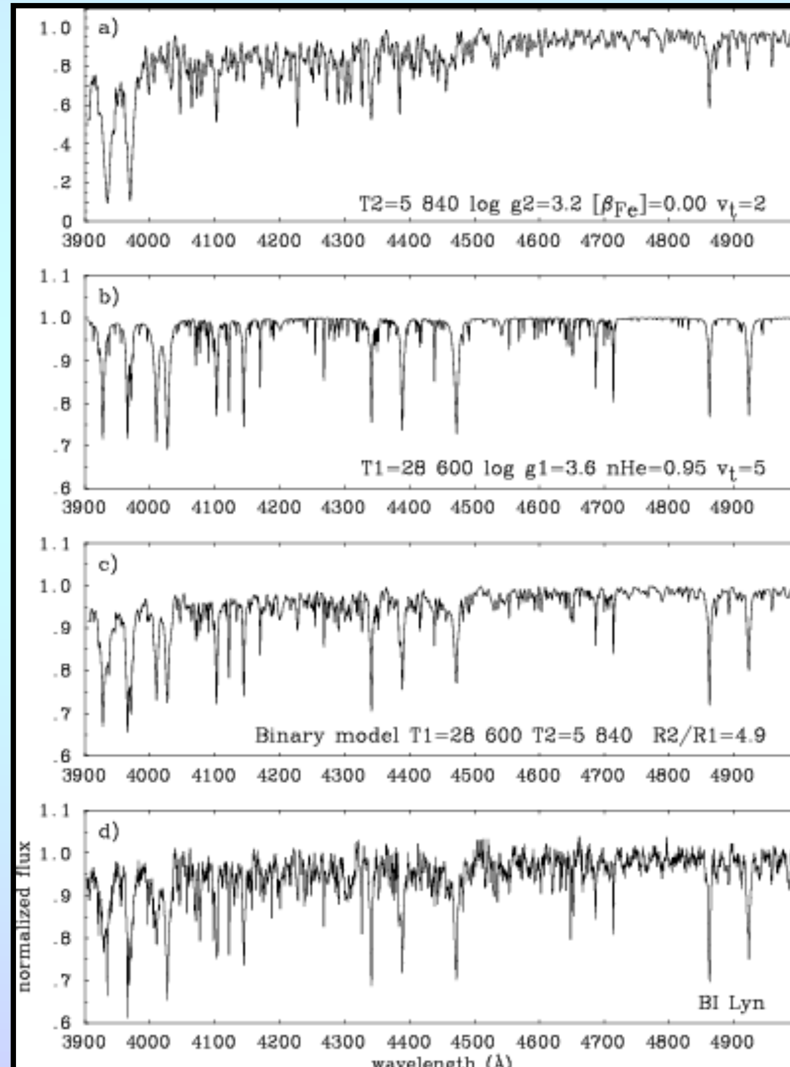



# Binary Stars

Most direct measurements of stellar radii and stellar masses come from the analysis of binary stars. There are several types of binaries:

- Visual
- Spectrum

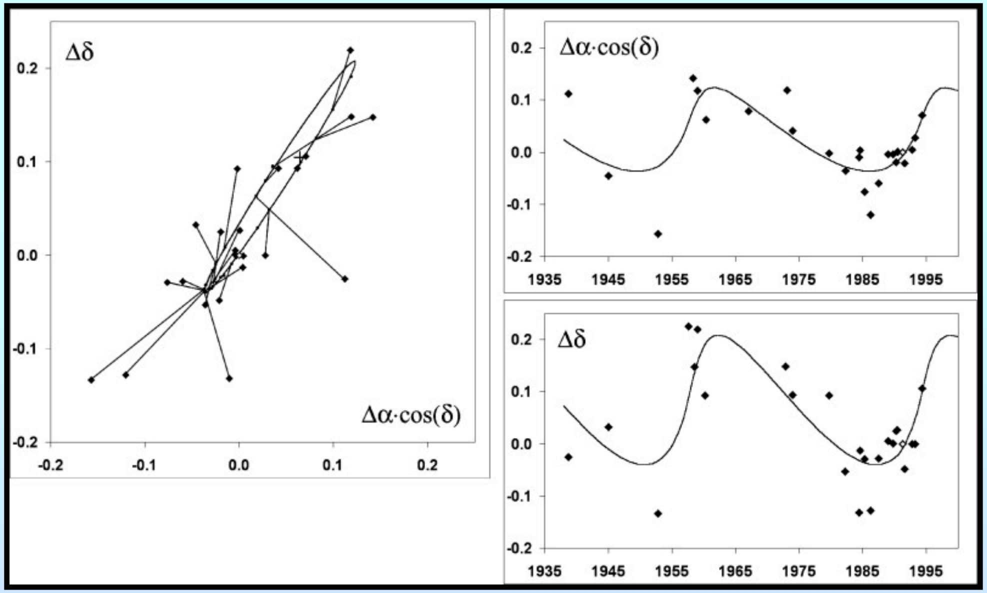


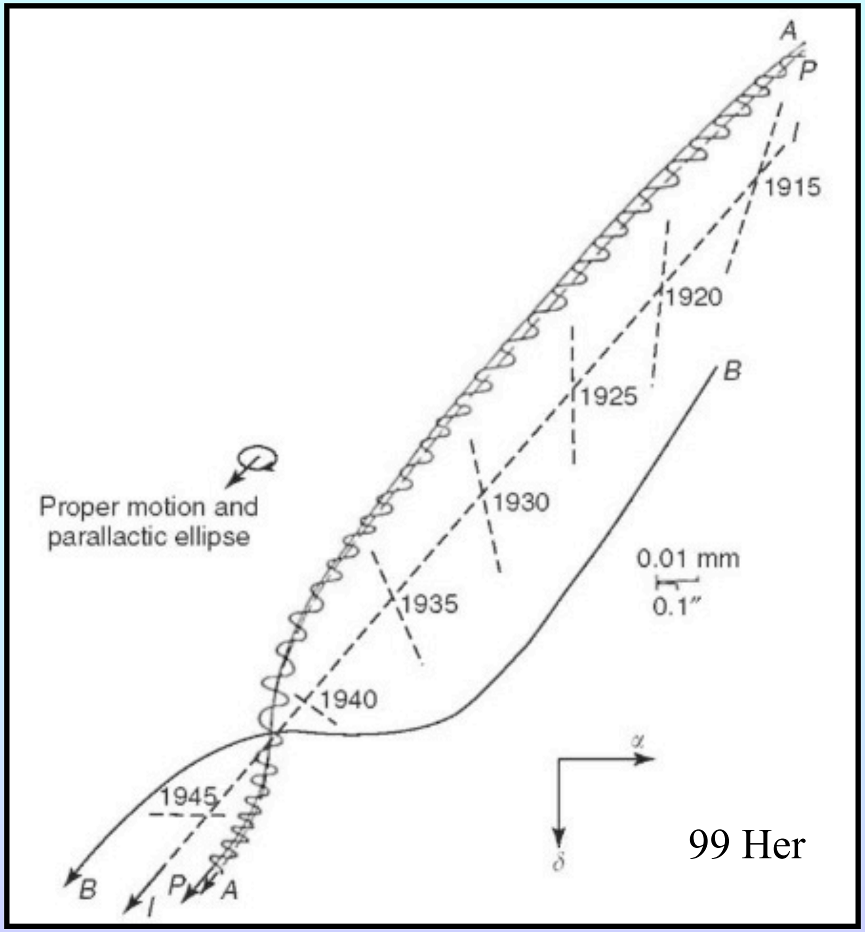



# Binary Stars

Most direct measurements of stellar radii and stellar masses come from the analysis of binary stars. There are several types of binaries:

- Visual
- Spectrum
- Astrometric

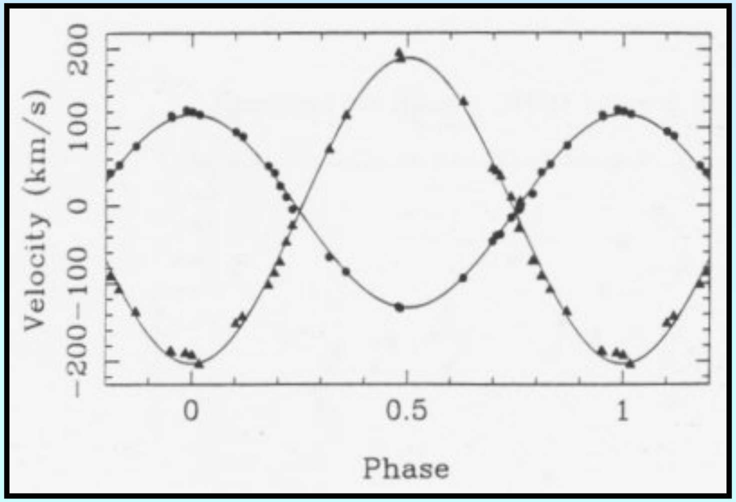


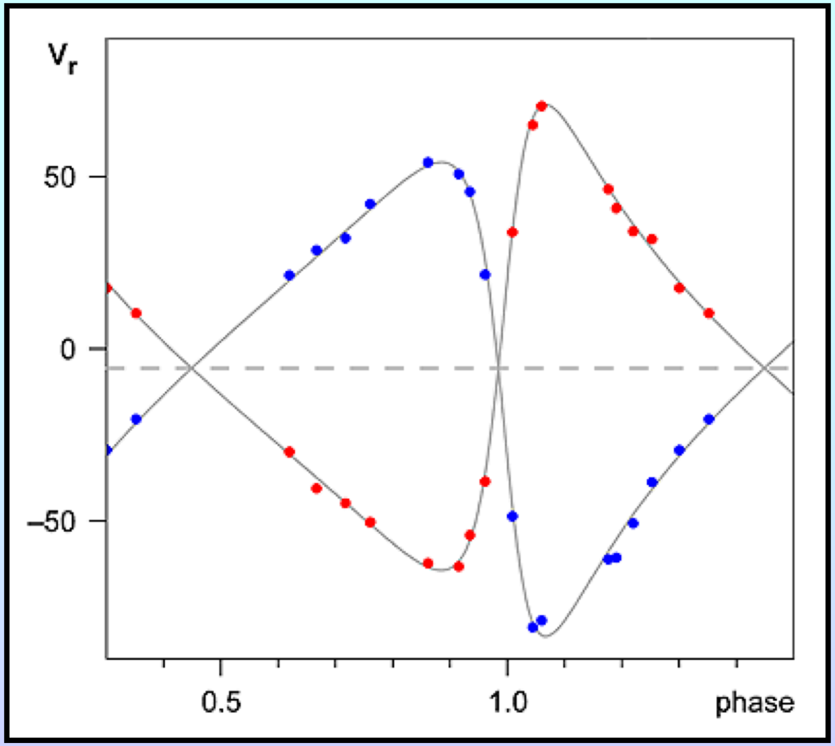



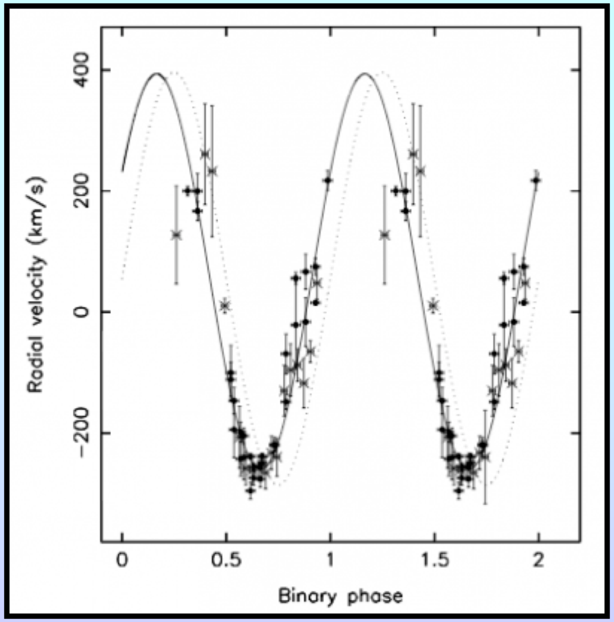
# Binary Stars

Most direct measurements of stellar radii and stellar masses come from the analysis of binary stars. There are several types of binaries:

- Visual
- Spectrum
- Astrometric
- Spectroscopic



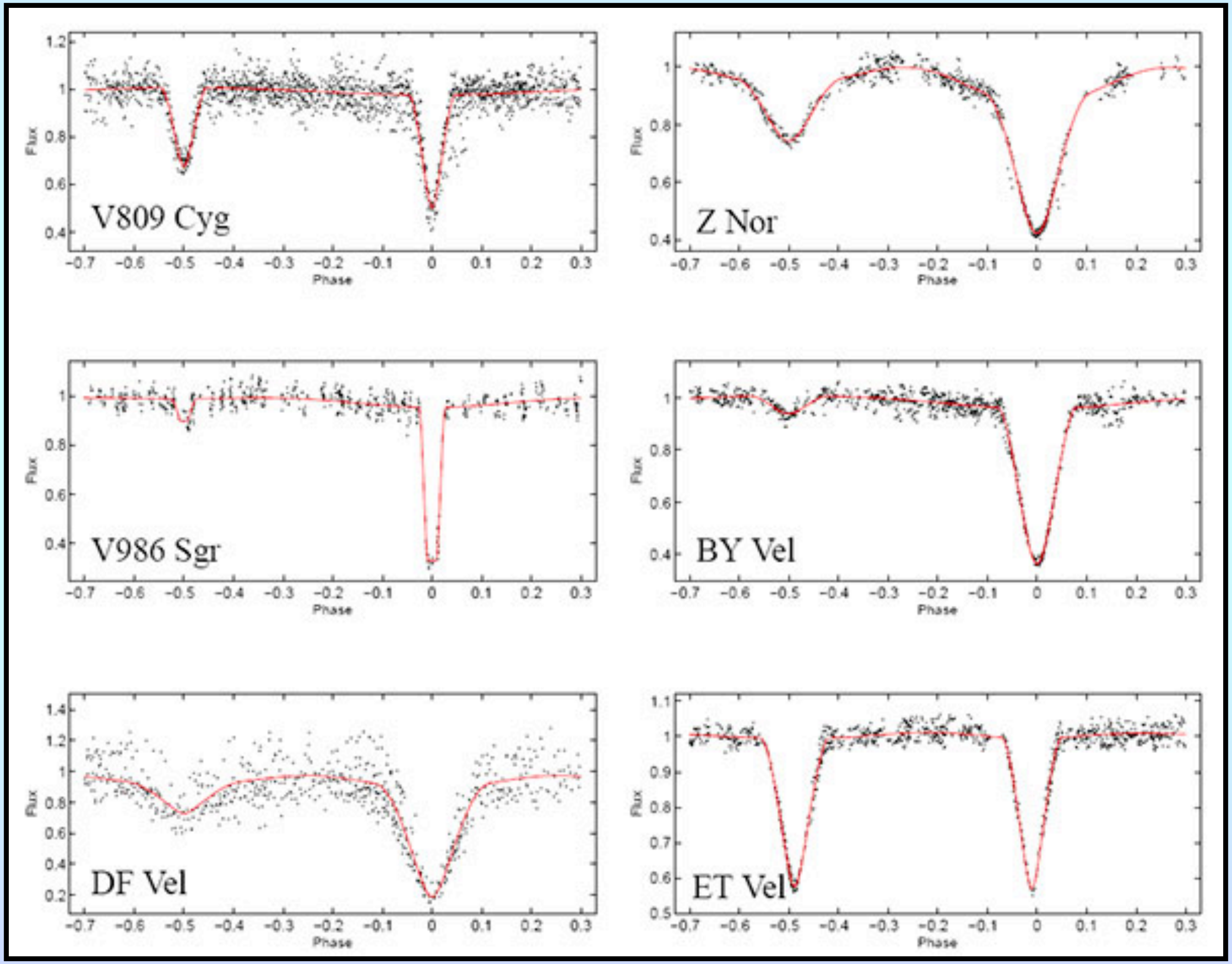



# Binary Stars

Most direct measurements of stellar radii and stellar masses come from the analysis of binary stars. There are several types of binaries:

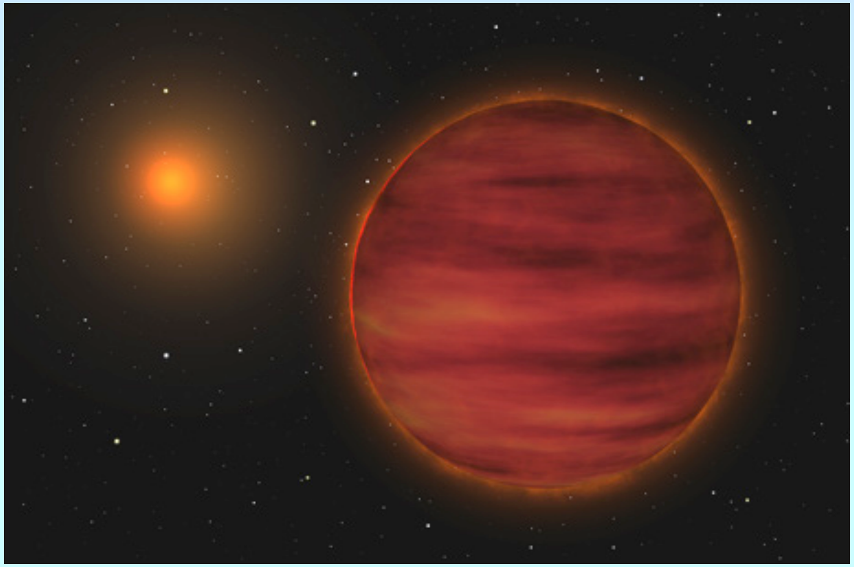
- Visual
- Spectrum
- Astrometric
- Spectroscopic
- Eclipsing

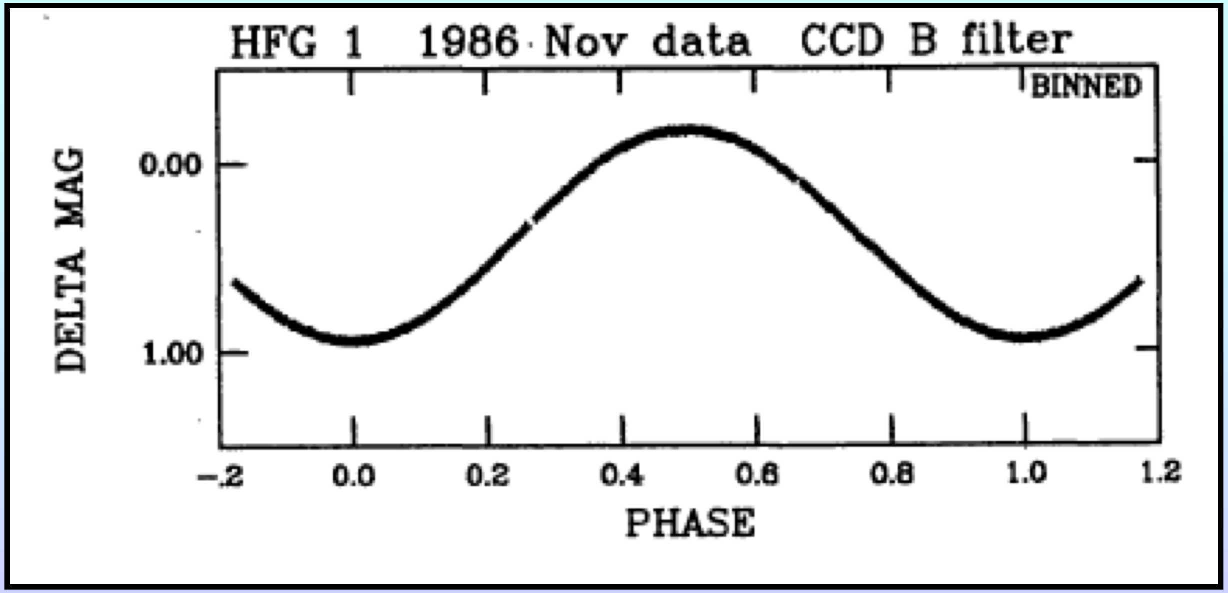


# Binary Stars

Most direct measurements of stellar radii and stellar masses come from the analysis of binary stars. There are several types of binaries:

- Visual
- Spectrum
- Astrometric
- Spectroscopic
- Eclipsing
- Reflection

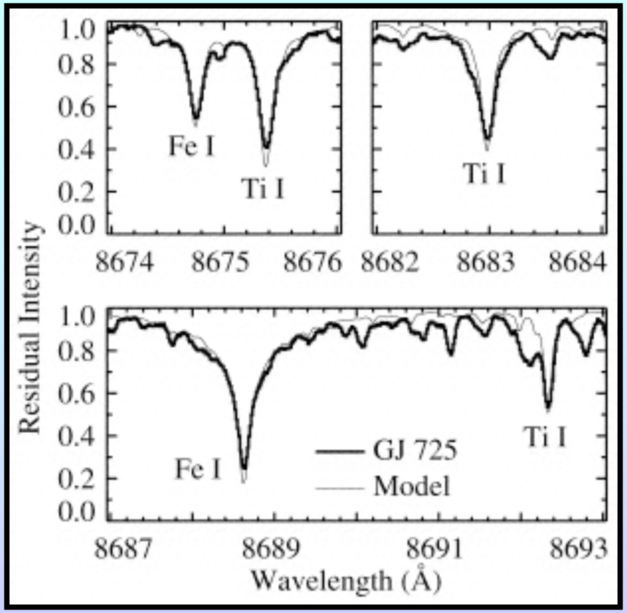


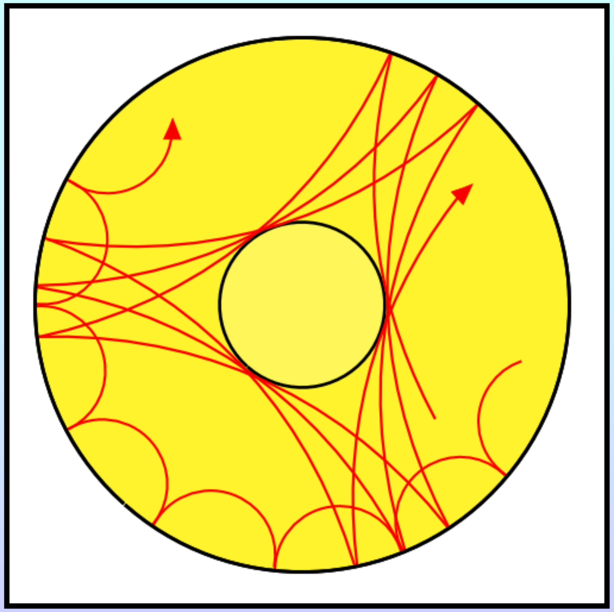


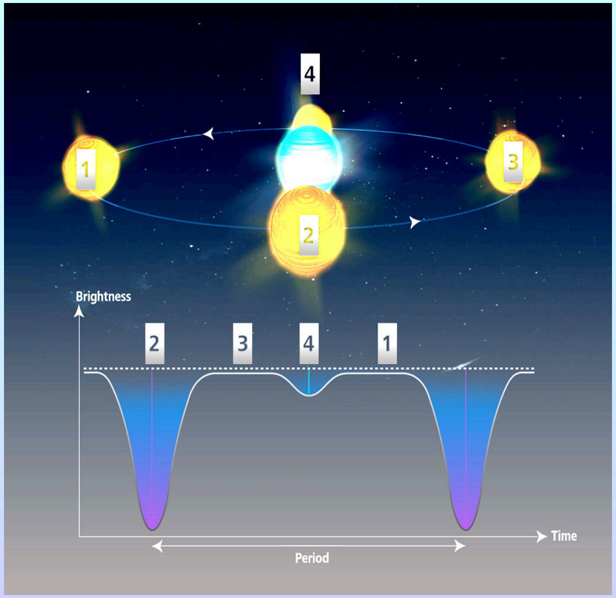
# Obtaining Stellar Masses

There are basically 3 techniques that can be used to measure stellar masses:

- Analysis of stellar absorption lines (pressure broadening, etc.) One measures (log) $g = G\,M\,/\,R^2 = 4\,\pi\,\sigma\,G\,M\,T^4/\,L$
- Asteroseismology of non-radial pulsators
- Binary Stars





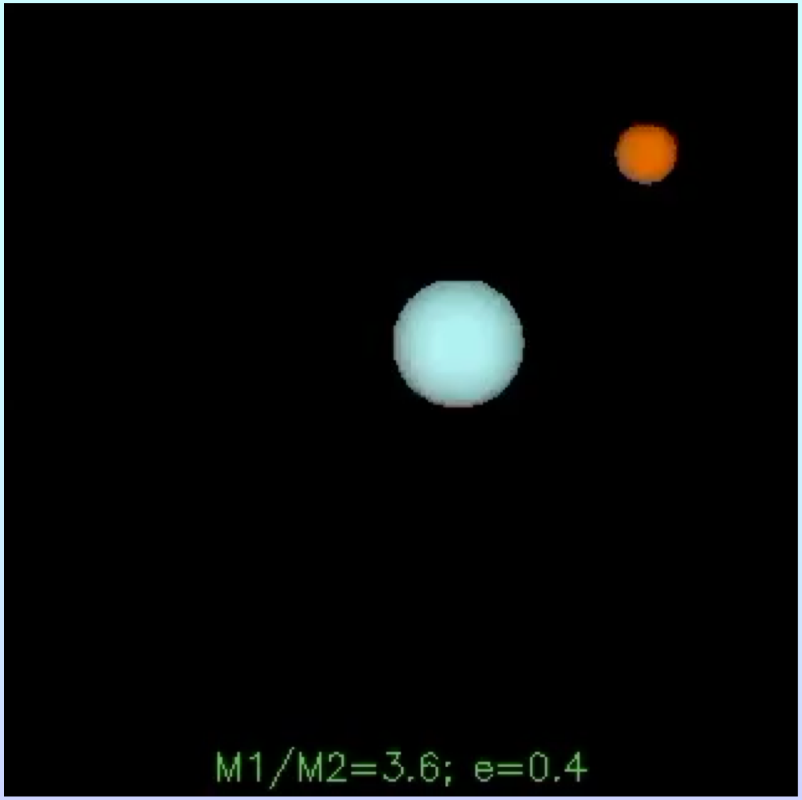
# The Motions of Binary Stars

- The motions of binary stars are controlled by Kepler's laws:
  - The orbits are ellipses with a star at a focus
  - The orbits sweep out equal areas in equal times
  - $(M_1 + M_2)\,P^2 = a^3$

# The Motions of Binary Stars

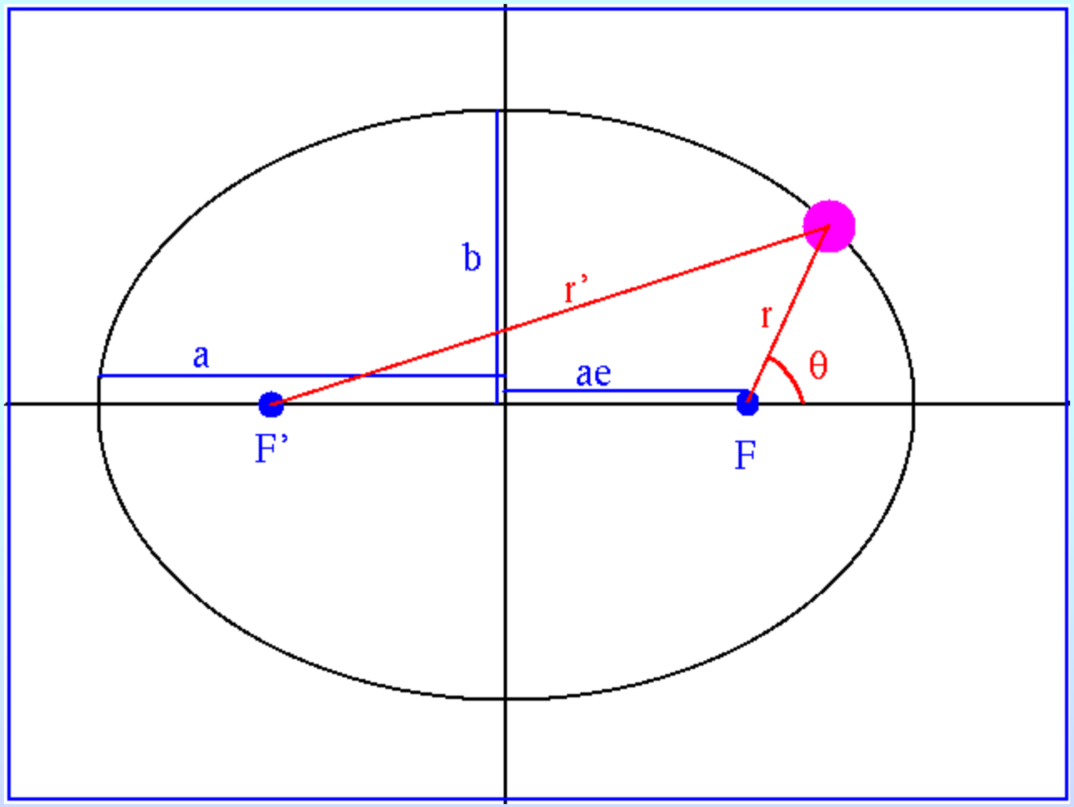
- The motions of binary stars are controlled by Kepler's laws:
  - The orbits are ellipses with a star at a focus
  - The orbits sweep out equal areas in equal times
  - $(M_1 + M_2)\, P^2 = a^3$
- The separations and motions of the two stars in the center-of-mass frame are given by

$$\boxed{\frac{M_1}{M_2} = \frac{a_2}{a_1} = \frac{r_2(t)}{r_1(t)} = \frac{v_2(t)}{v_1(t)}}$$



# Kepler's First Law

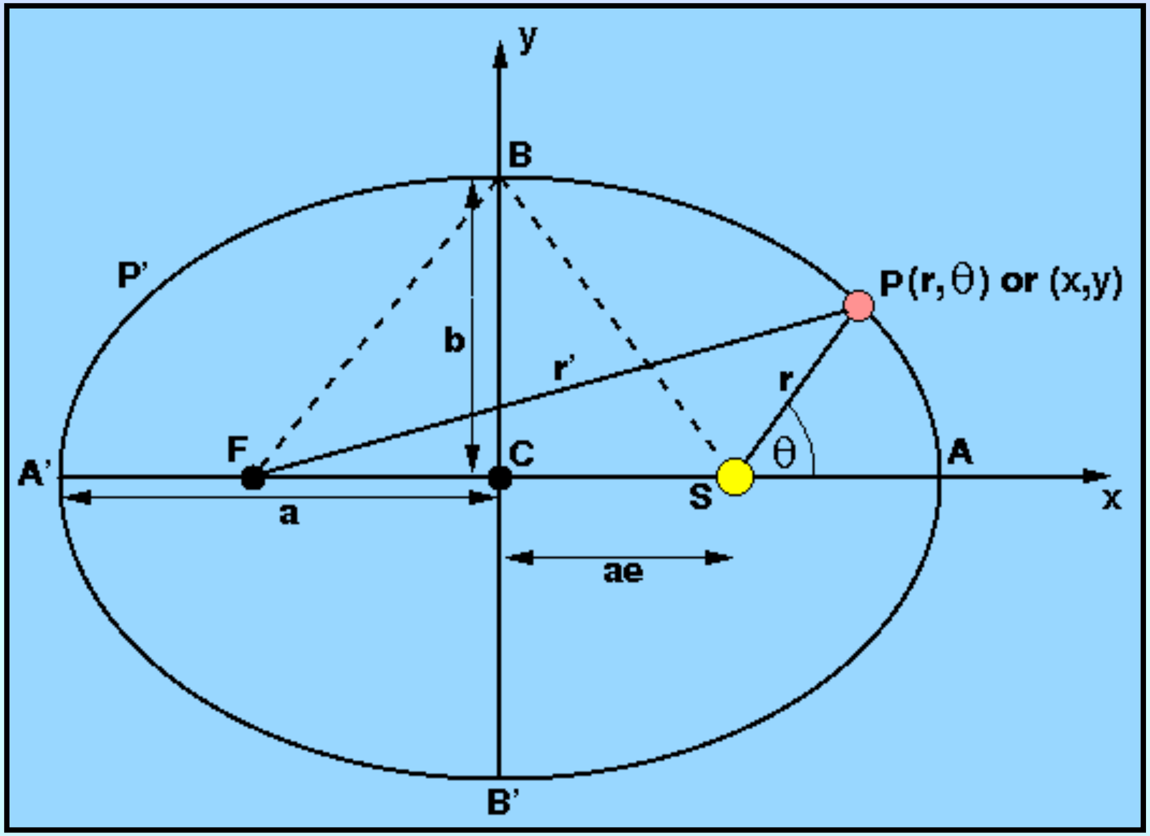
The orbit of one body, with respect to the other, is a conic section with one of the bodies at a focus. For bound systems, the conic system is an ellipse. (An ellipse is a shape defined by 2 focii; for any point on the ellipse, the sum of the distances to the 2 focii is constant.)




- $a$ = semi-major axis
- $b$ = semi-minor axis
- $b/a$ = ellipticity
- ε = eccentricity

$$\varepsilon = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

# Ellipses




Distance from center to focus of ellipse: $a$ ε

True Anomaly: θ

Periapsis: $a$ (1 – ε)

Apapsis: $a$ (1 + ε)

# Equations of an Ellipse

Cartesian coordinates:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Polar coordinates (more useful):

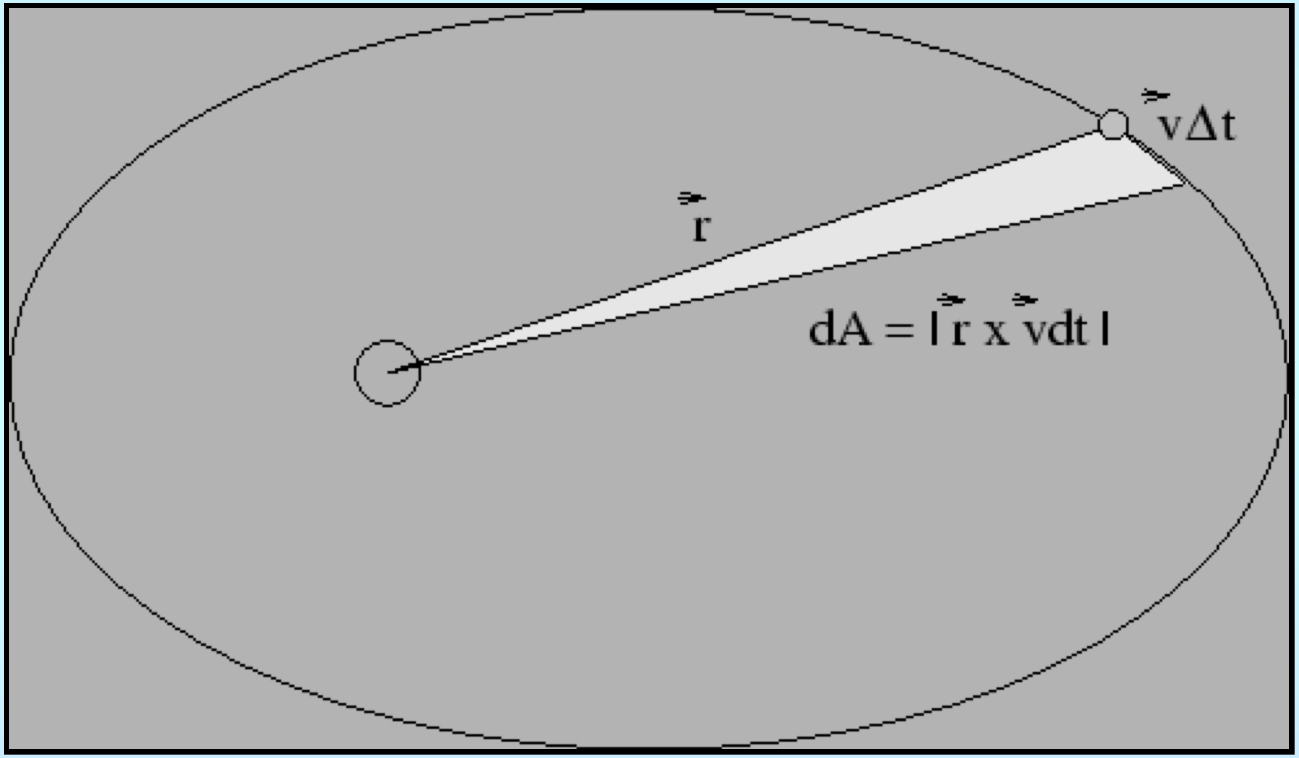
$$r = \frac{a\left(1 - \varepsilon^2\right)}{1 + \varepsilon \cos\theta}$$

External parameterization (handy for graphics):

$$x = a \cos\xi \qquad y = b \sin\xi$$

# Kepler's Second Law

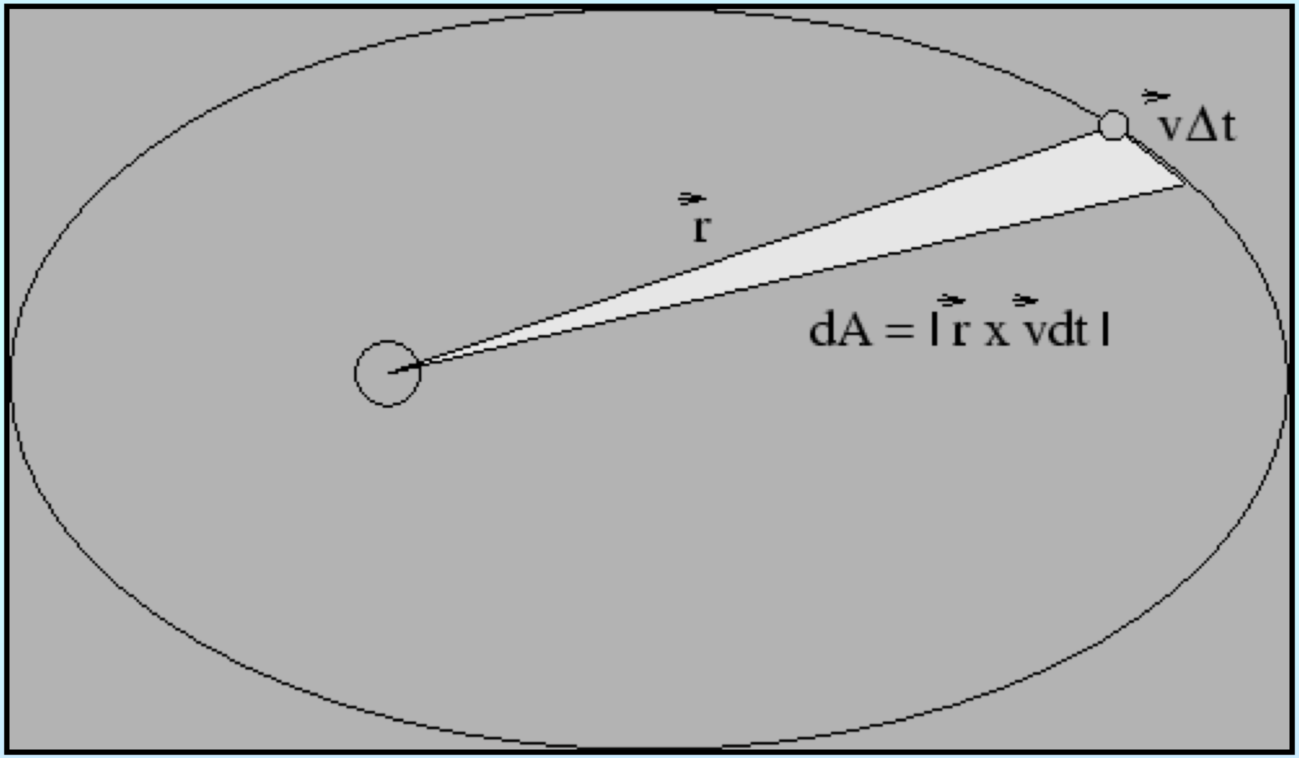
The line connecting the two bodies sweeps out a constant amount of area per unit time. (In other words, the system conserves angular momentum.) The angular momentum per unit mass is




$$dA = \frac{1}{2}\left|\vec{\mathbf{r}} \times \vec{\mathbf{v}}\,\mathrm{dt}\right| = \frac{1}{2m}\left|\vec{\mathbf{r}} \times m\vec{\mathbf{v}}\,dt\right| = \frac{1}{2m}\left|\vec{\mathbf{L}}\,dt\right|$$

# Kepler's Second Law

The line connecting the two bodies sweeps out a constant amount of area per unit time. (In other words, the system conserves angular momentum.) The angular momentum per unit mass is




$$\vec{\mathbf{r}} \times \vec{\mathbf{v}} = r^2 \frac{d\theta}{dt} = \frac{2\pi ab}{P} = \frac{2\pi a^2 \left(1-\varepsilon^2\right)^{1/2}}{P}$$

# Kepler's 3rd Law (as modified by Newton)

The orbital period squared times the total mass of the system is proportional to the mean separation of the two bodies cubed.

$$a^3 = k\,P^2\left(M_1 + M_2\right)$$

where $k = G/4\pi^2$. Note, however, that if distance is measured in A.U., period in years, and mass in solar masses, then from scaling to the Earth-Sun system, $k = 1$.

In any binary, the *brighter* of two stars is called the primary. It is usually (but not necessarily) also the most massive.

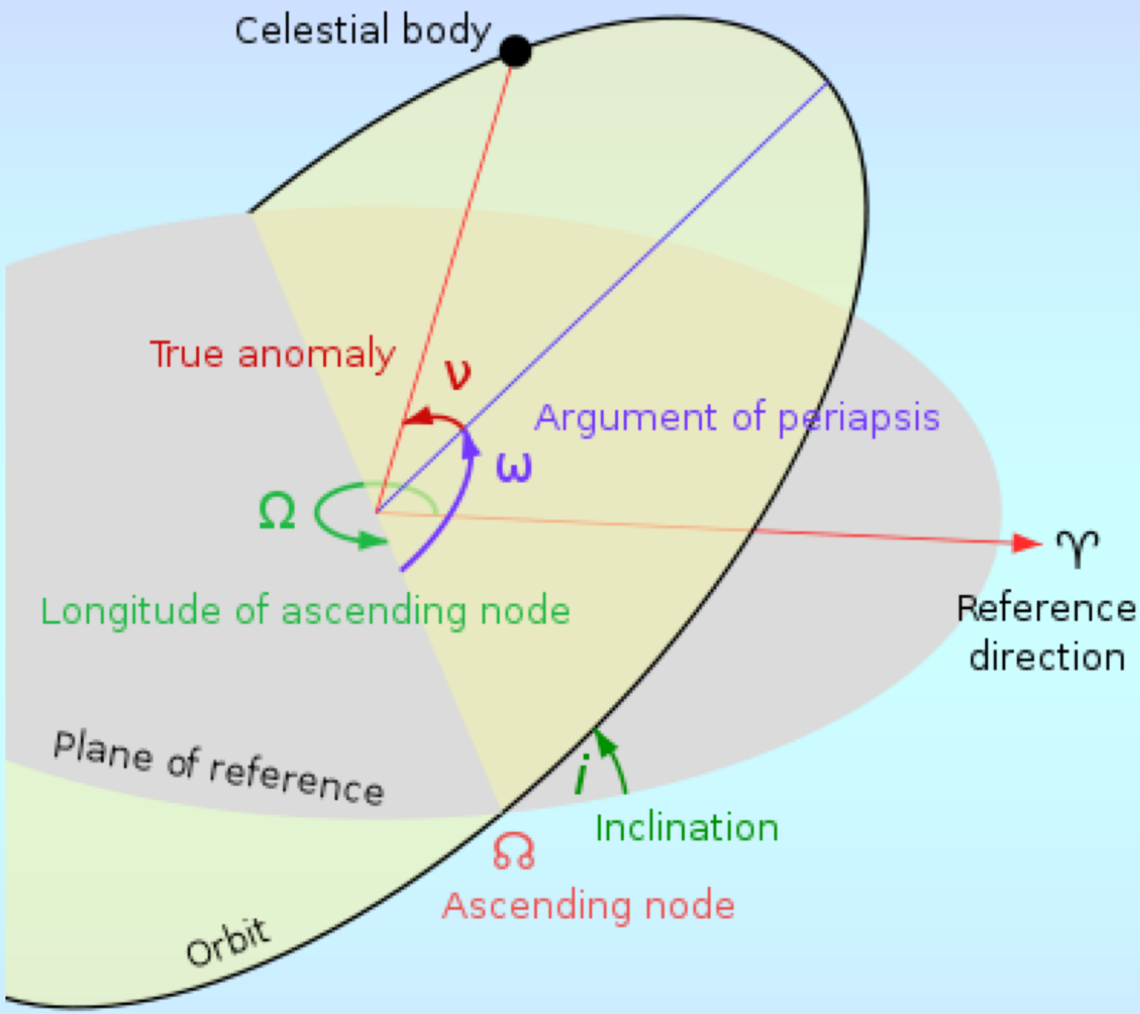
# Defining an Orbit

Seven (or six) orbital elements define a body's orbit

- The shape and size of the ellipse:
  - The semi-major axis ($a$)
  - The eccentricity ($\varepsilon$)
- The direction of the plane of the ellipse:
  - The inclination ($i$)
  - Longitude of ascending node ($\Omega$)
- The orientation of the ellipse's major axis:
  - The argument of periapsis ($\omega$)
- The location and speed of the object:
  - Time of periapsis ($T_p$) or mean anomaly at epoch ($M_0$)
  - The period ($P$) if not in the Solar System

In the Solar System, $a$ implies $P$ (since the mass is known). You therefore only have 6 variables (and thus, at minimum, you need 6 observations to define an orbit).

# Defining an Orbit




Note: The position of an orbiting body is most easily described in polar coordinates via an angle. But usually, you want position versus time.

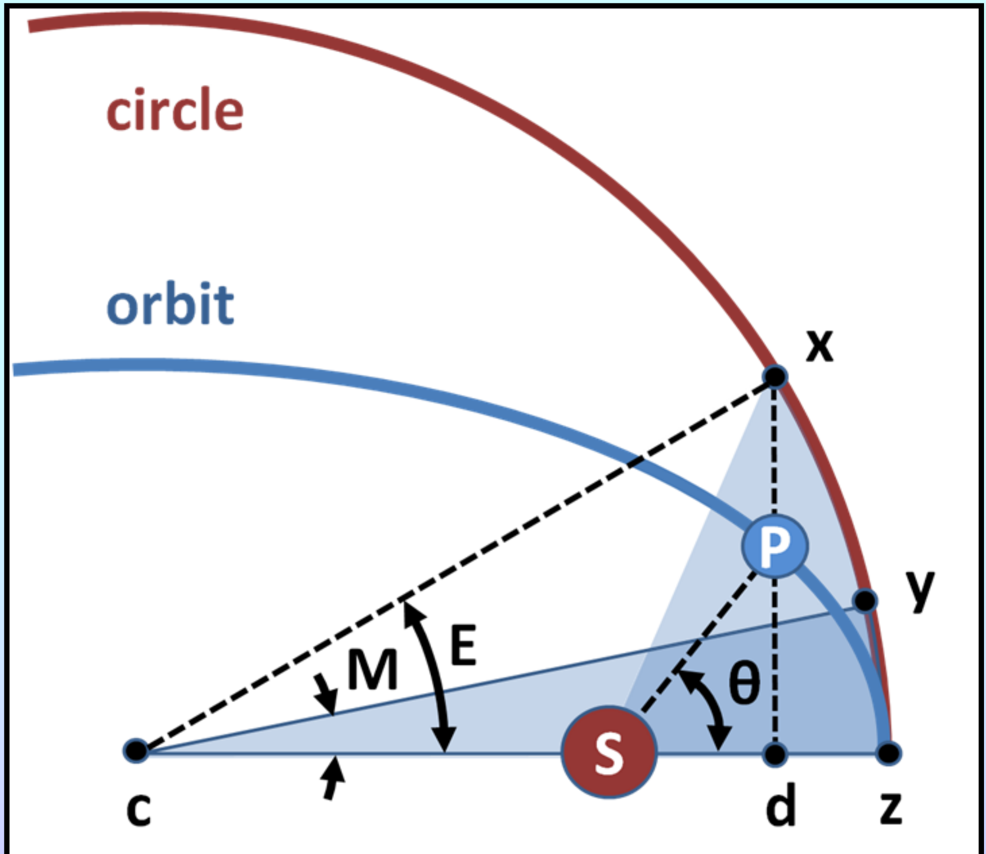
# Relating Angles to Time

An "anomaly" is the angle with respect to the periapsis.

- The "true anomaly" ($\theta$) is the angle to the object from the focus.
- The "mean anomaly" ($M$) is the angle from the *center* of the ellipse to a fictitious object traveling in a circular orbit with radius, $a$, and period, $P$, i.e.,

$$M = \frac{2\pi}{P}\left(t - T_p\right)$$

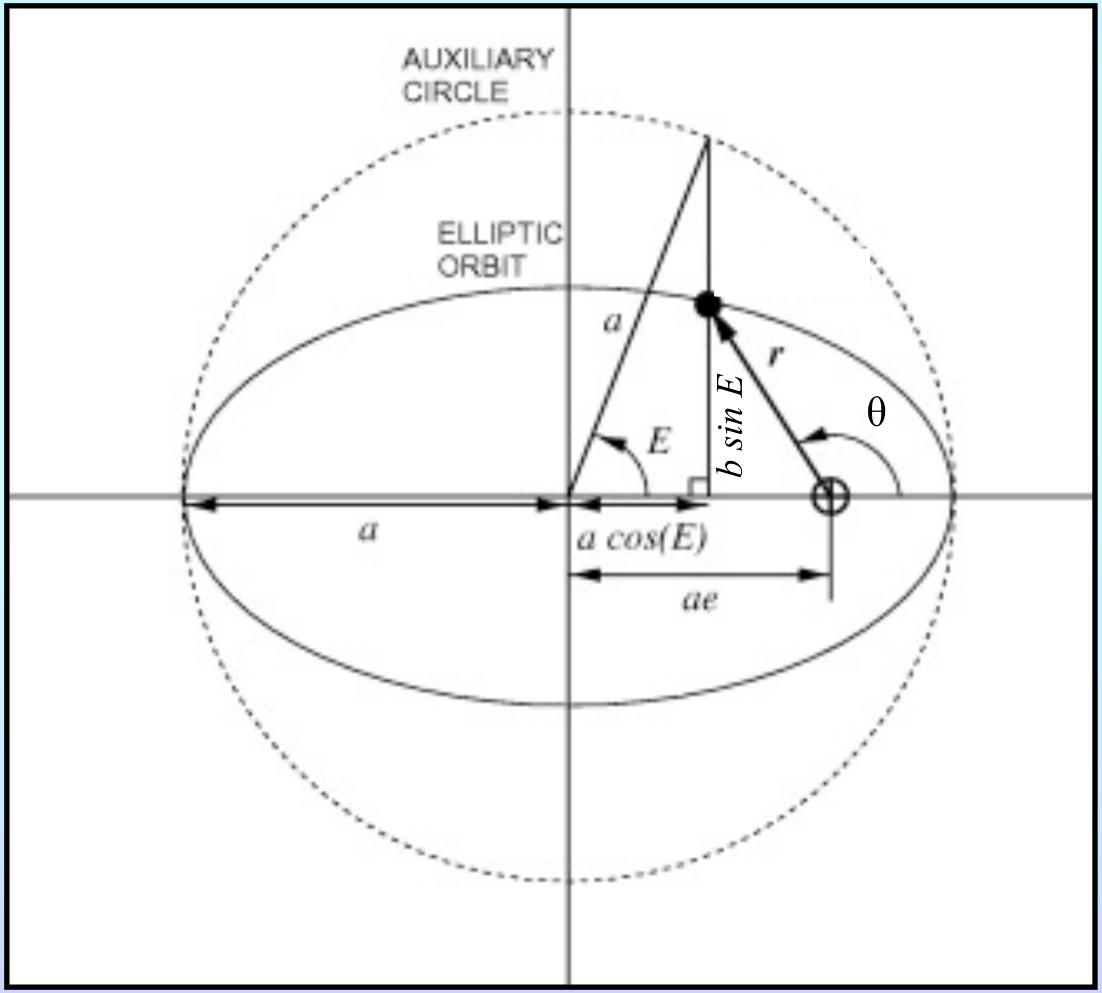
- The "eccentric anomaly" ($E$) is the angle from the *center* of the ellipse to the point on a circle with radius $a$ which intersects the perpendicular from the semi-major axis though the object.




$t(\theta)$ is non analytic, but $t(M)$ is trivial. $t(E)$ connects the two.

# Relating the True and Eccentric Anomaly

The true anomaly, $\theta$, can be related to the eccentric anomaly, *E,* using the geometry of triangles.




$$r = a\left(1 - \varepsilon\cos E\right)$$

$$\cos\theta = \frac{\cos E - \varepsilon}{1 - \varepsilon\cos E}$$

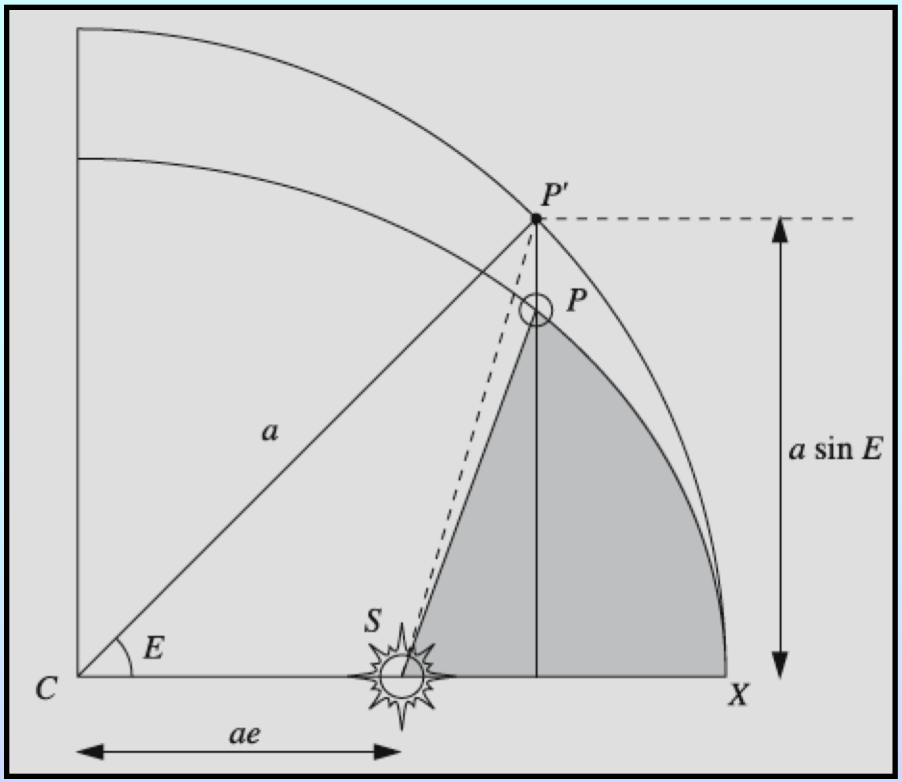
$$\sin\theta = \frac{\left(1 - \varepsilon^2\right)^{1/2}\sin E}{1 - \varepsilon\cos E}$$

$$\tan^2(\theta/2) = \frac{1+\varepsilon}{1-\varepsilon}\tan^2(E/2)$$

# Relating the Mean and Eccentric Anomaly

The eccentric anomaly can be computed by noting that the area of an ellipse sector is equal to that of a similar sector of a circumscribed circle, reduced by *b/a*. In other words

$$\text{Area(SPX)} = \frac{b}{a}\text{Area(SP}'\text{X)} = \frac{b}{a}\text{Area(CP}'\text{X)} - \text{Area}(\Delta\text{CP}'\text{S})$$




$$= \frac{b}{a}\left\{\left(\pi a^2 \cdot \frac{E}{2\pi}\right) - \frac{1}{2}(a\varepsilon)(a\sin E)\right\}$$

$$= \frac{ab}{2}\left(E - \varepsilon\sin E\right)$$

Since

$$\text{Area(SPX)} = \pi a b \frac{t - T_p}{P} = \frac{ab}{2} M$$

$$M = E - \varepsilon \sin E$$

This is Kepler's Equation

# Relating Time to Position

To obtain an object's position versus time

- Time can trivially be converted to the mean anomaly

$$M = \frac{2\pi}{P}\left(t - T_p\right)$$

- Mean anomaly can be converted to eccentric anomaly via Kepler's equation

$$M = E - \varepsilon \sin E$$

- The eccentric anomaly can be translated into the true anomaly via geometric relations, i.e.,

$$\tan^2(\theta/2) = \frac{1+\varepsilon}{1-\varepsilon}\tan^2(E/2)$$

- Finally, the radius vector is found from the true anomaly using

$$r = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta}$$

# Binary Star Velocities

Note the combination of Kepler's 1st and 2nd law defines the velocity of the binary at any moment. For instance, at the orbit's apapsis, $r = a\ (1 + \varepsilon)$, and the radius and velocity vectors are perpendicular, so sin $\varphi = 1$ and

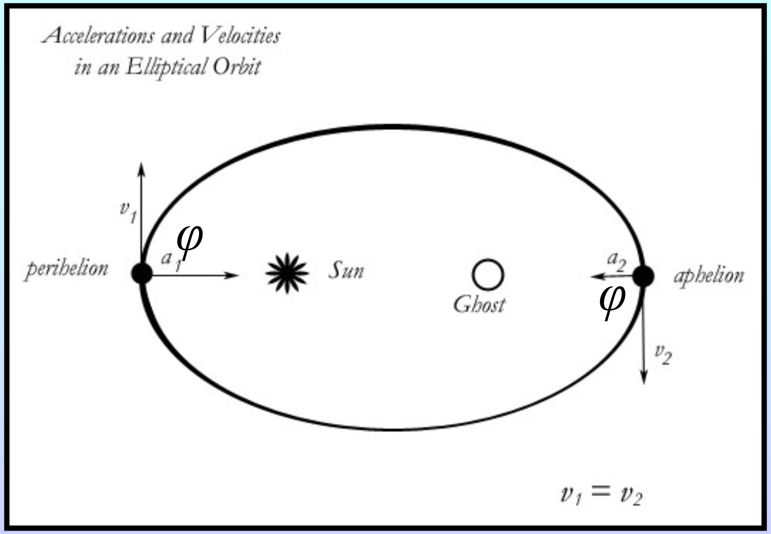
$$r^2 \frac{d\theta}{dt} = r v_{\mathrm{ap}} \sin\varphi = r v_{\mathrm{ap}} = a(1+\varepsilon) v_{\mathrm{ap}} = \frac{2\pi a^2 \left(1-\varepsilon^2\right)^{1/2}}{P}$$

So Kepler's 3rd law gives

$$v_{\mathrm{ap}} = \frac{2\pi a}{P}\sqrt{\frac{1-\varepsilon}{1+\varepsilon}} = \sqrt{\frac{G(M_1+M_2)}{a}}\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}$$

Similarly, at periapsis

$$v_{\mathrm{peri}} = \frac{2\pi a}{P}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}} = \sqrt{\frac{G(M_1+M_2)}{a}}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}$$



# Vis Viva Equation

To obtain the velocity at any point in the orbit, just use energy conservation:

$$\frac{1}{2}v^2 - \frac{GM_T}{r} = \frac{1}{2}v_{\mathrm{peri}}^2 - \frac{GM_T}{r_{\mathrm{peri}}} = \frac{1}{2}\left(\frac{GM_T}{a}\right)\left(\frac{1+\varepsilon}{1-\varepsilon}\right) - \frac{GM_T}{a(1-\varepsilon)}$$

$$= \frac{GM_T}{a}\frac{1}{1-\varepsilon}\left(\frac{1+\varepsilon}{2} - 1\right)$$

$$= -\frac{GM_T}{2a}\left(\frac{1-\varepsilon}{1-\varepsilon}\right) = -\frac{GM_T}{2a}$$

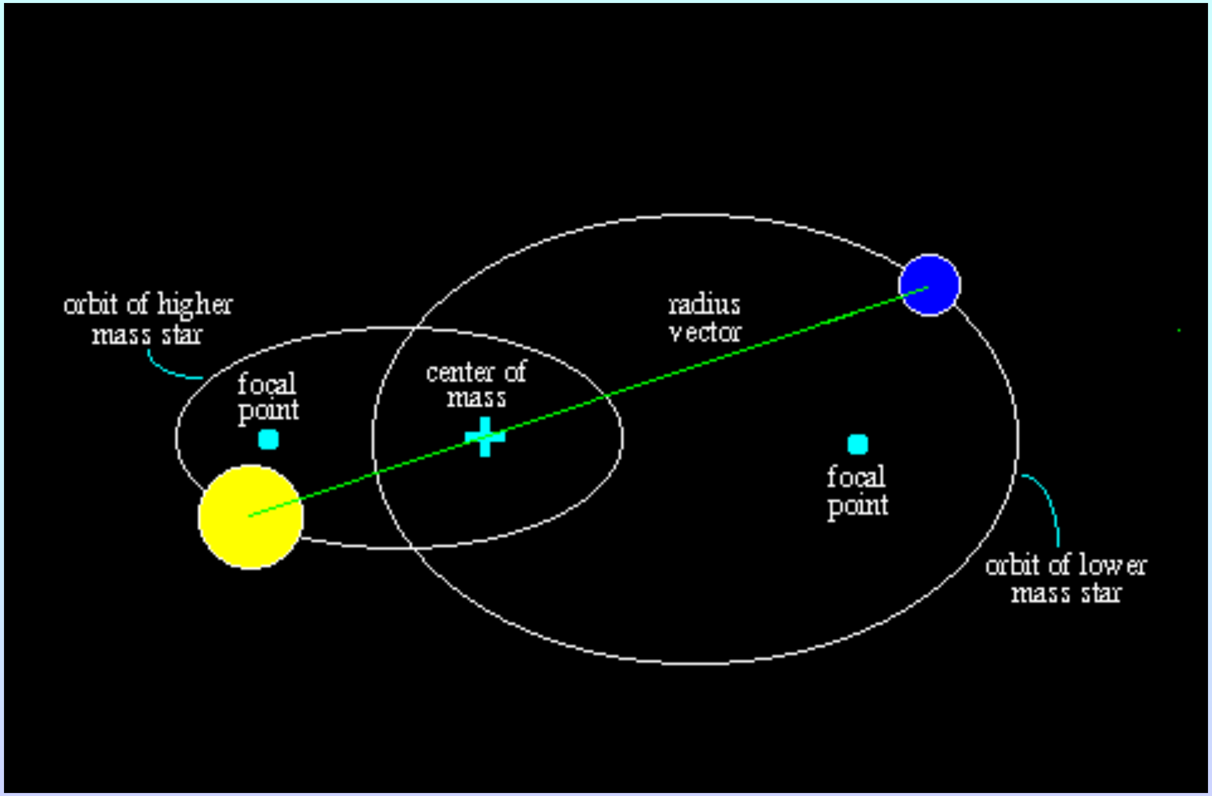
so

$$v^2(r) = G\left(M_1 + M_2\right)\left(\frac{2}{r} - \frac{1}{a}\right) \quad \text{where} \quad r = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta}$$

# Visual Binaries

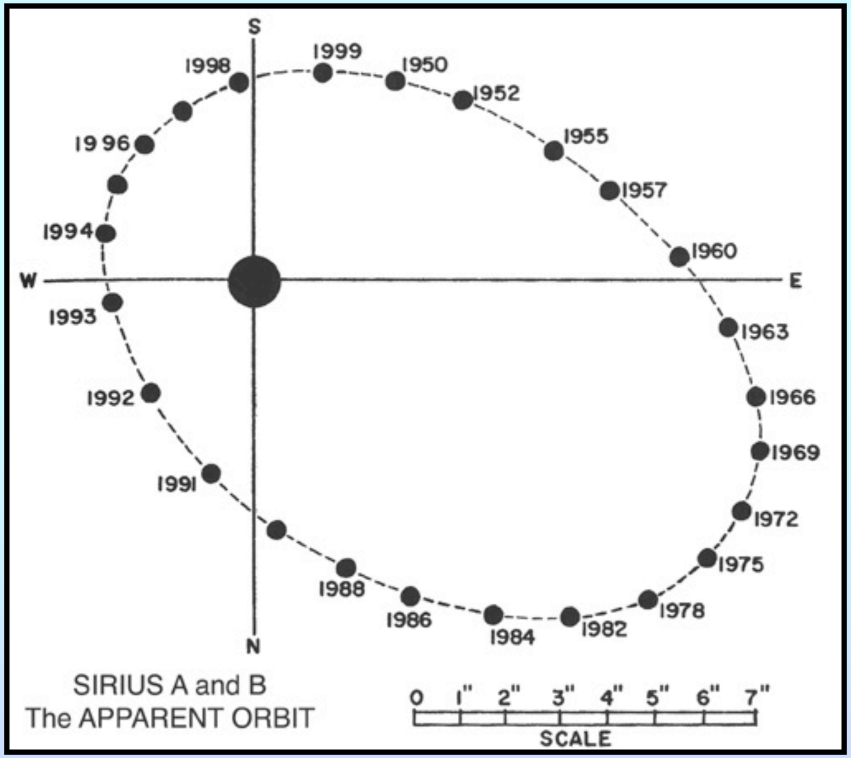
A perfect mass estimate of both stars is possible if:

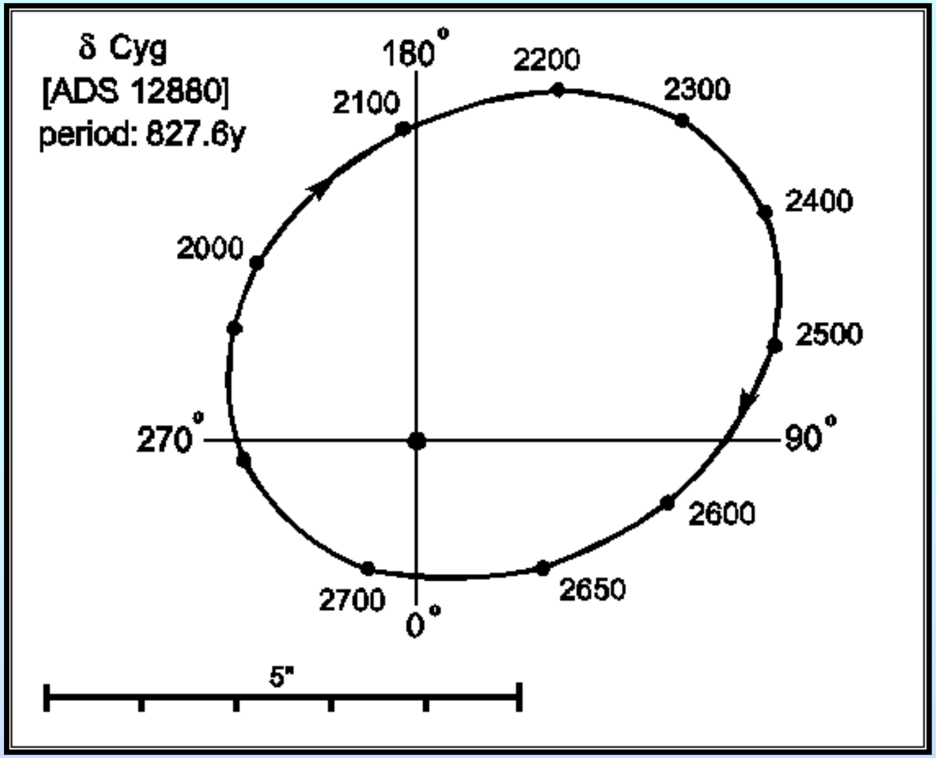
- Both stars are visible
- Their angular velocity is sufficiently high to allow a reasonable fraction of the orbit to be mapped
- The distance to the system is known (e.g., via parallax)



# Periods of Visual Binaries

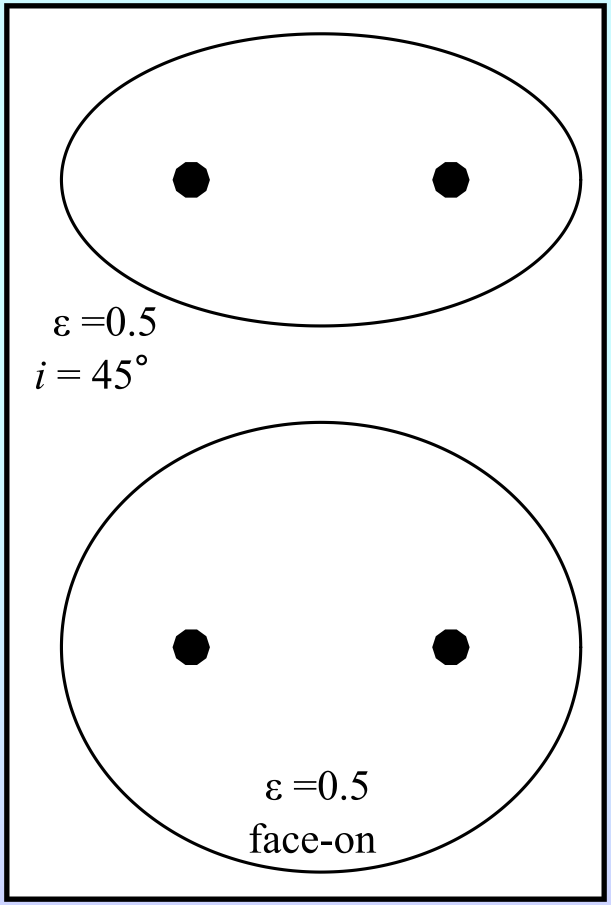
Note that for a solar-type visual binary at 10 pc, an $\alpha = 1''$ separation corresponds to a physical separation of 10 A.U. and a period of 33 years. More generally, for a distance $d$, $P \propto d^{3/2}\,\alpha^{3/2}$. So the periods can be *very* long.

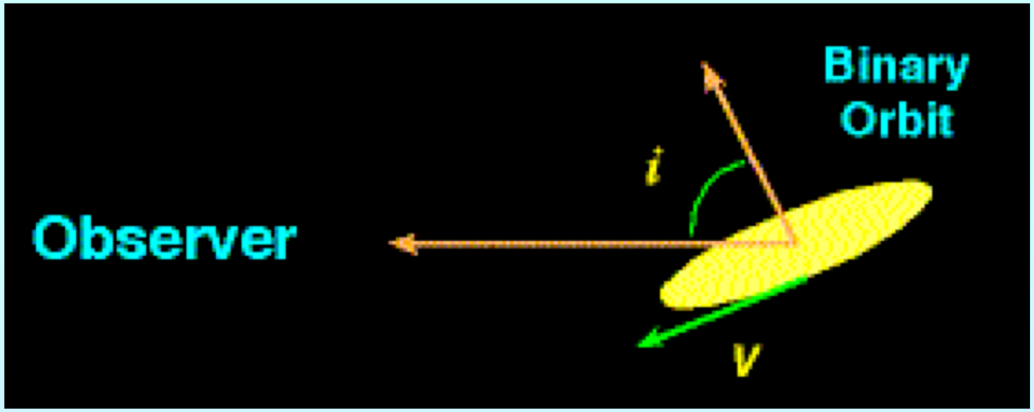


# Visual Binaries

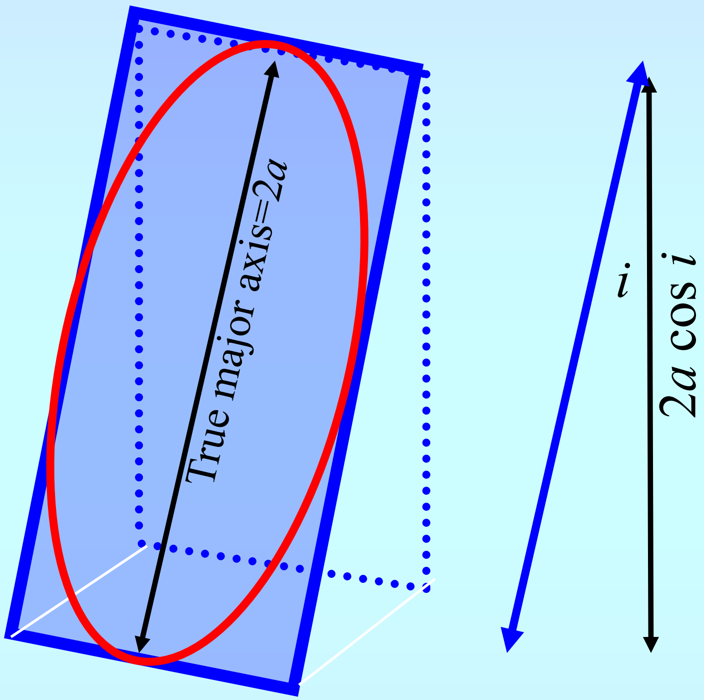
One difficulty with visual binaries is that, in general, you don't know the orbital inclination. (Is the orbit elliptical or an inclined circle?) There is the true ellipse and an apparent ellipse, but the focus of the true ellipse is not the focus of the apparent ellipse.

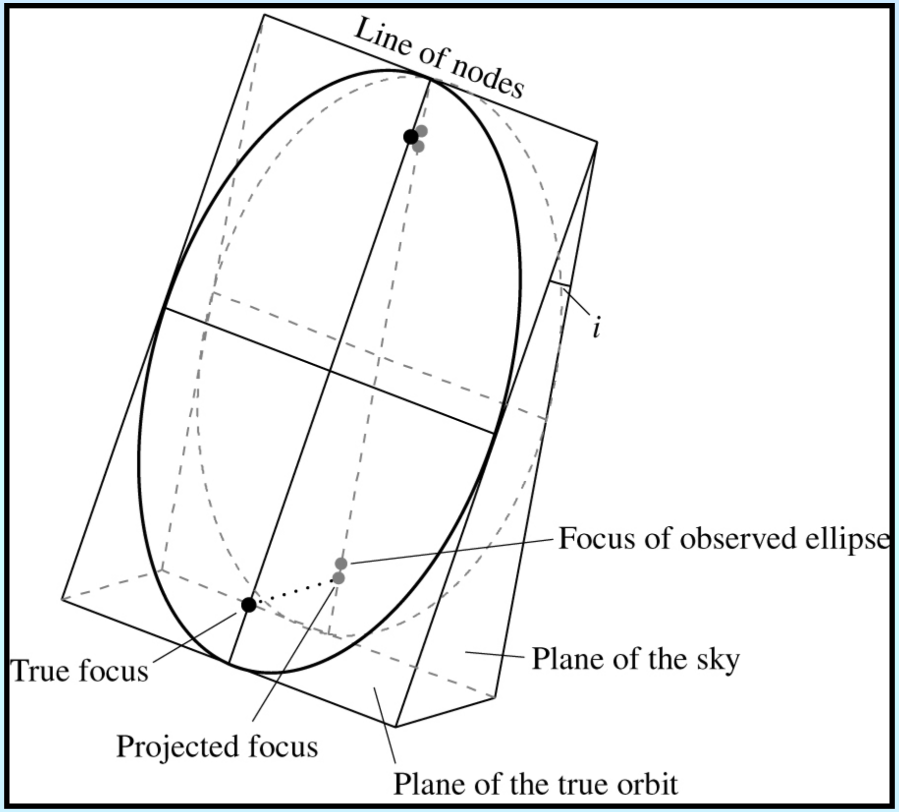



For visual binaries, you measure $a \cos i$

# Masses from Visual Binaries




The projection distorts the ellipse: the center of mass is not at the observed focus and the observed eccentricity is not the true eccentricity. Only with long-term, precise observations can you determine the true orbit.

# Masses from Visual Binaries

The observed angular separation of a binary at a distance $d$ will be $\alpha' = d\,\alpha_0 \cos i$. Relative masses will be unaffected by inclination

$$\boxed{\frac{M_1}{M_2} = \frac{a_2}{a_1} = \frac{r_2}{r_1} = \frac{d\alpha_2}{d\alpha_1} = \frac{d\alpha_2'/\cos i}{d\alpha_1'/\cos i} = \frac{\alpha_2}{\alpha_1}}$$

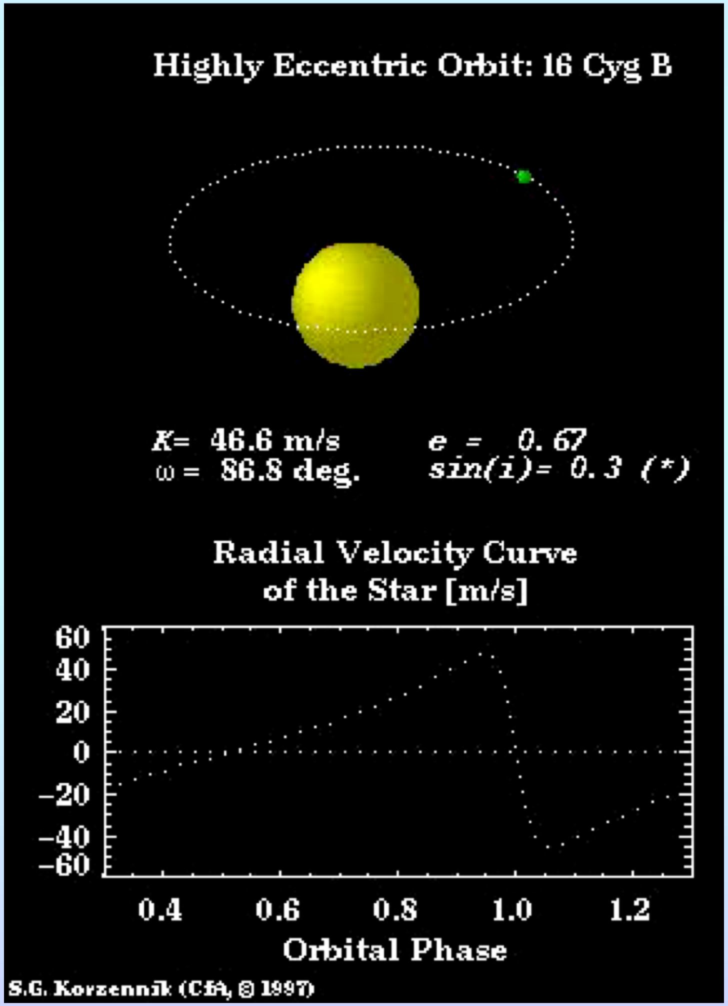
However, our measurement of total mass will only be a lower limit.

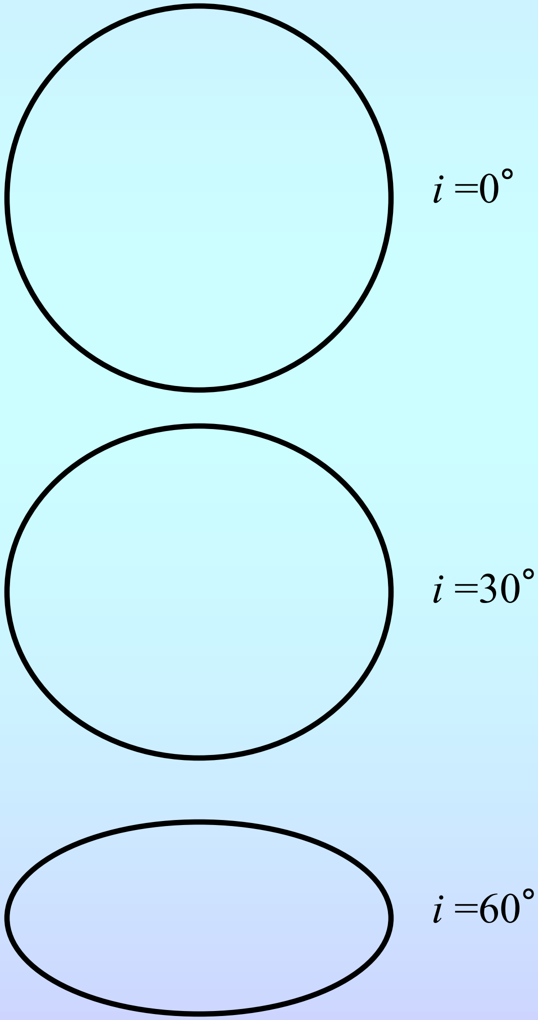
$$M_1 + M_2 = \frac{a^3}{k\,P^2} = \frac{(d\alpha')^3}{k\,P^2} = \frac{(d\alpha_0/\cos i)^3}{k\,P^2} = \left(\frac{d}{\cos i}\right)^3 \frac{\alpha_0^3}{k\,P^2}$$

The masses derived from visual binaries will depend on distance cubed and $\cos^3 i$.

# Spectroscopic Binaries

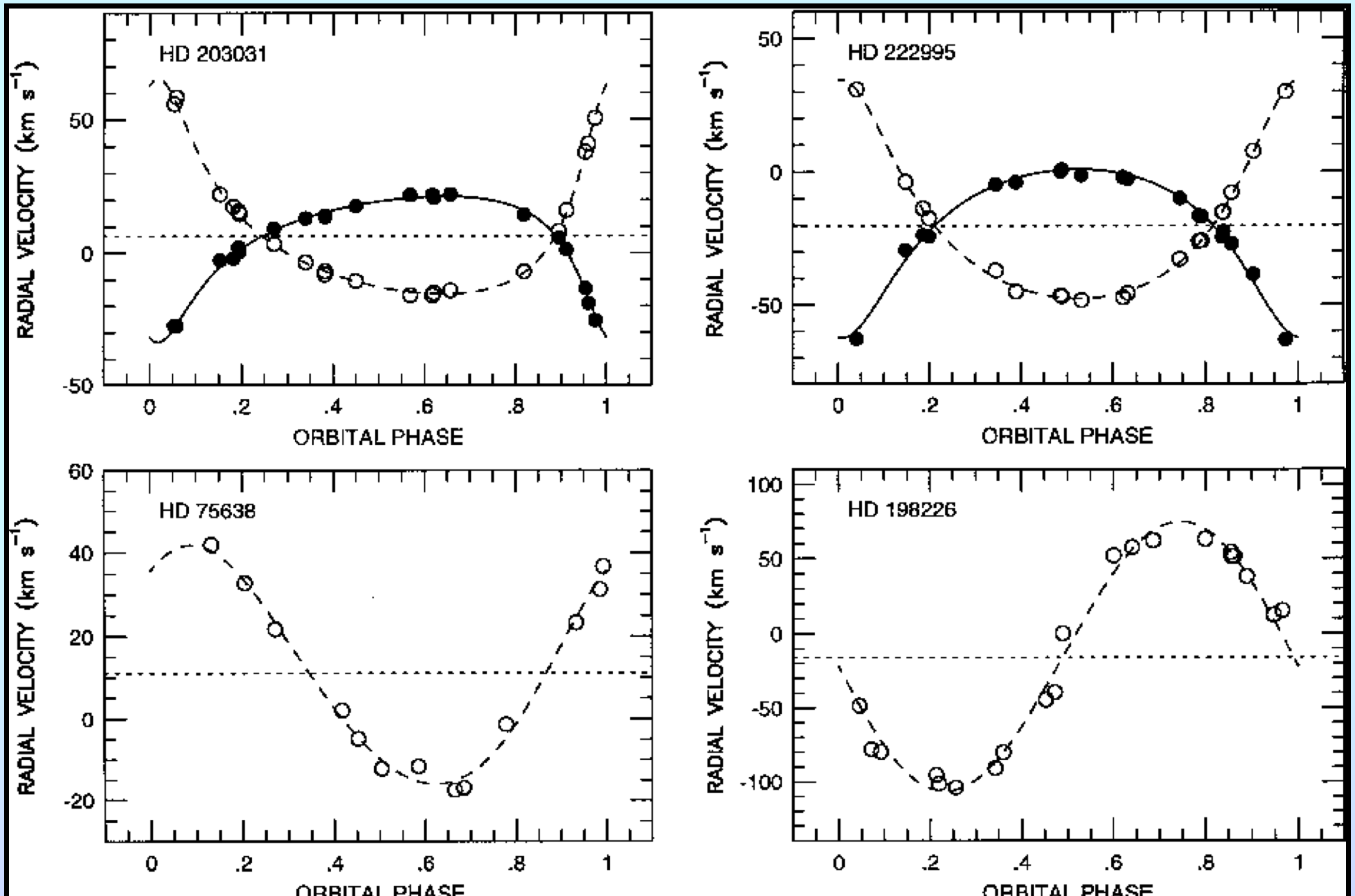
Orbits in the plane of the sky ($i = 0°$ ) show no radial velocity. In general, $v_{obs} = v_{true} \sin i$.



# Separations of Spectroscopic Binaries

For a solar-type star at ~ 10 pc, a ~ 1 year period corresponds to a separation of 0.1″, and a 1 km/s velocity corresponds to ~ 5 A.U. Most spectroscopic binaries are therefore unresolved.



# Spectroscopic Binaries

Spectroscopic binaries have the opposite problem as visual binaries: we can observe stellar motions in the radial direction, but not in the plane of the sky. If $z$ is the (Cartesian) line-of-sight coordinate, then at any time, the $z$-position of a star will be

$$z = r\sin(\omega+\theta)\sin i \quad \text{where} \quad r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}$$

The star's radial velocity will therefore be

$$v_z = \dot{z} = \sin i\left\{\dot{r}\sin(\omega+\theta) + r\dot{\theta}\cos(\omega+\theta)\right\}$$

After some manipulation, this becomes

$$v_z = \gamma + K\left\{\cos(\omega+\theta) + \varepsilon\cos\omega\right\} \quad \text{with} \quad K = \frac{2\pi}{P}\frac{a\sin i}{\sqrt{1-\varepsilon^2}}$$

$K_1$ and $K_2$ are the semi-amplitudes of the radial velocity curves of the primary and the secondary, and $\gamma$ (the gamma velocity) is the systemic velocity of the system.

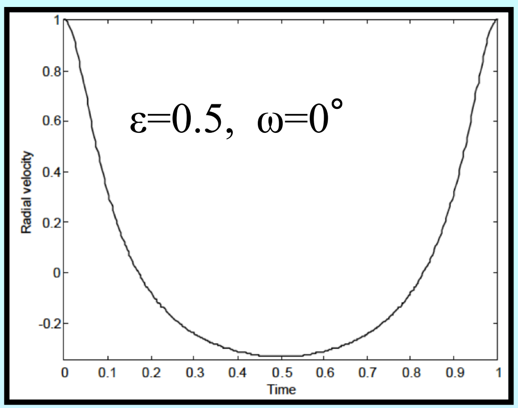
# Spectroscopic Binaries

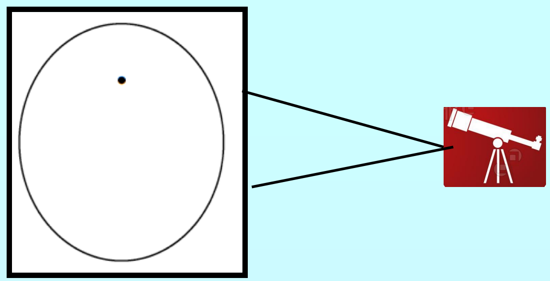
Note that the maximum and minimum measured velocities will be

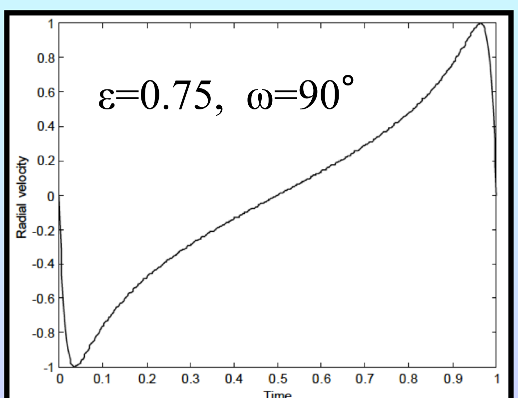
$$v_{\max} = \gamma + K(\varepsilon \cos\omega + 1) \quad \text{and} \quad v_{\min} = \gamma + K(\varepsilon \cos\omega - 1)$$

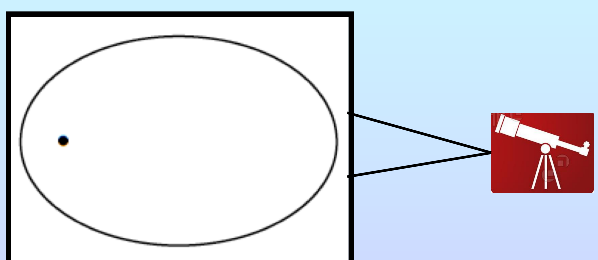
so

$$K = \frac{v_{\max} - v_{\min}}{2} \quad \text{and} \quad \varepsilon \cos\omega = \frac{v_{\max} + v_{\min}}{2K} \quad \text{with} \quad K = \frac{2\pi}{P}\frac{a \sin i}{\sqrt{1-\varepsilon^2}}$$





From the shape and amplitude of the velocity curve, one can measure ε, ω, $P$, and $T_p$. But one can't separate $a$ from sin $i$.

# Single and Double Line Binaries

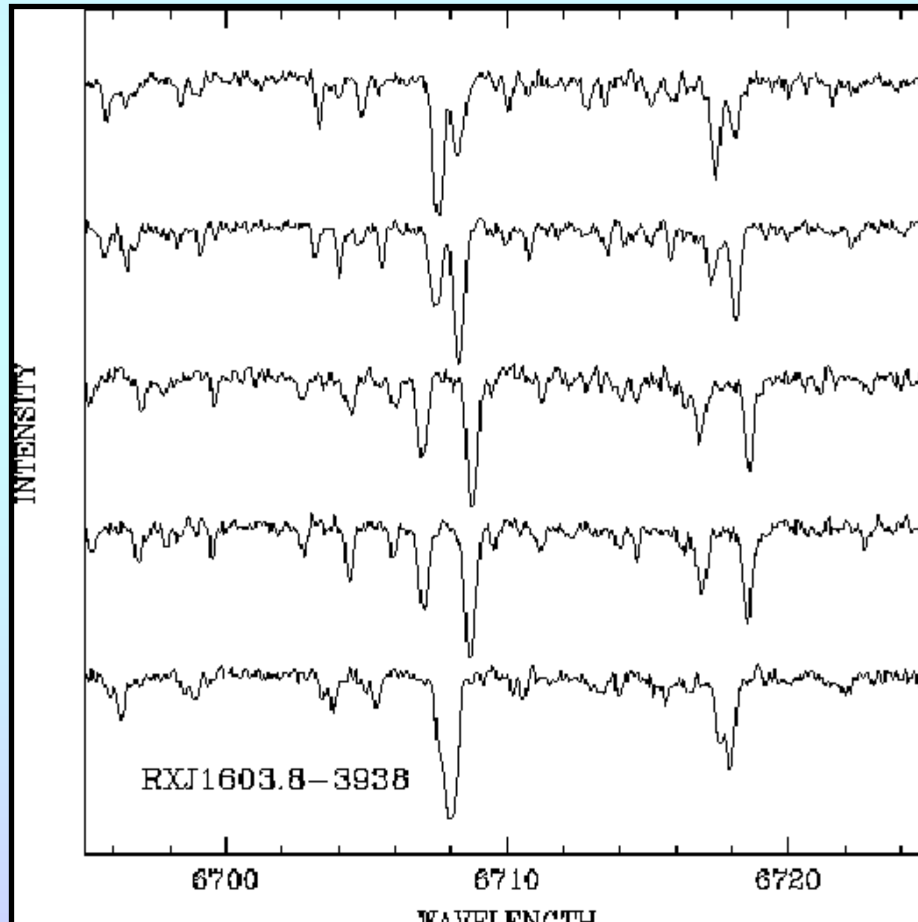
If one star is much brighter than the other, the spectrum from the secondary star may not be visible. You will have a single line spectroscopic binary. If both sets of lines are visible, it is called a double line spectroscopic binary.

Clearly, from the definition of the center of mass,

$$\frac{a_1}{a_2} = \frac{r_1}{r_2} = \frac{v_1}{v_2} = \frac{K_1}{K_2} = \frac{M_2}{M_1} = q$$

and since $a = a_1 + a_2$, we obtain equations like

$$a = a_1\left(\frac{M_1 + M_2}{M_2}\right)$$



# The Mass Function

For single line binaries, it is impossible to obtain the ratio of the stellar masses. But one can write down a function which depends on the *minimum mass* that the unseen companion must have. Start with Kepler's 3rd law:

$$M_1 + M_2 = \frac{a^3}{k P^2}$$

and substitute in the primary's semi-major axis using

$$a = a_1 \left( \frac{M_1 + M_2}{M_2} \right)$$

Kepler's 3rd law then becomes

$$M_1 + M_2 = \left( \frac{M_1 + M_2}{M_2} \right)^3 \frac{a_1^3}{k P^2} \quad \Rightarrow \quad \frac{M_2^3}{(M_1 + M_2)^2} = \frac{a_1^3}{k P^2}$$

Finally, since the semi-major axis isn't observable, substitute the $K_1$ velocity for $a_1$ using

$$K_1 = \frac{2\pi}{P} \frac{a_1 \sin i}{\sqrt{1 - \varepsilon^2}}$$

# The Mass Function

Kepler's 3rd law is then

$$\frac{M_2^3}{\left(M_1+M_2\right)^2}=\frac{a_1^3}{k\,P^2}=\frac{P^3K_1^3\left(1-\varepsilon^2\right)^{3/2}}{\left(2\pi\right)^3\sin^3 i}\cdot\frac{1}{k\,P^2}$$

or

$$f\left(M_2\right)=\frac{M_2^3\sin^3 i}{\left(M_2+M_1\right)^2}=\frac{P\,K_1^3\left(1-\varepsilon^2\right)^{3/2}}{k\left(2\pi\right)^3}$$

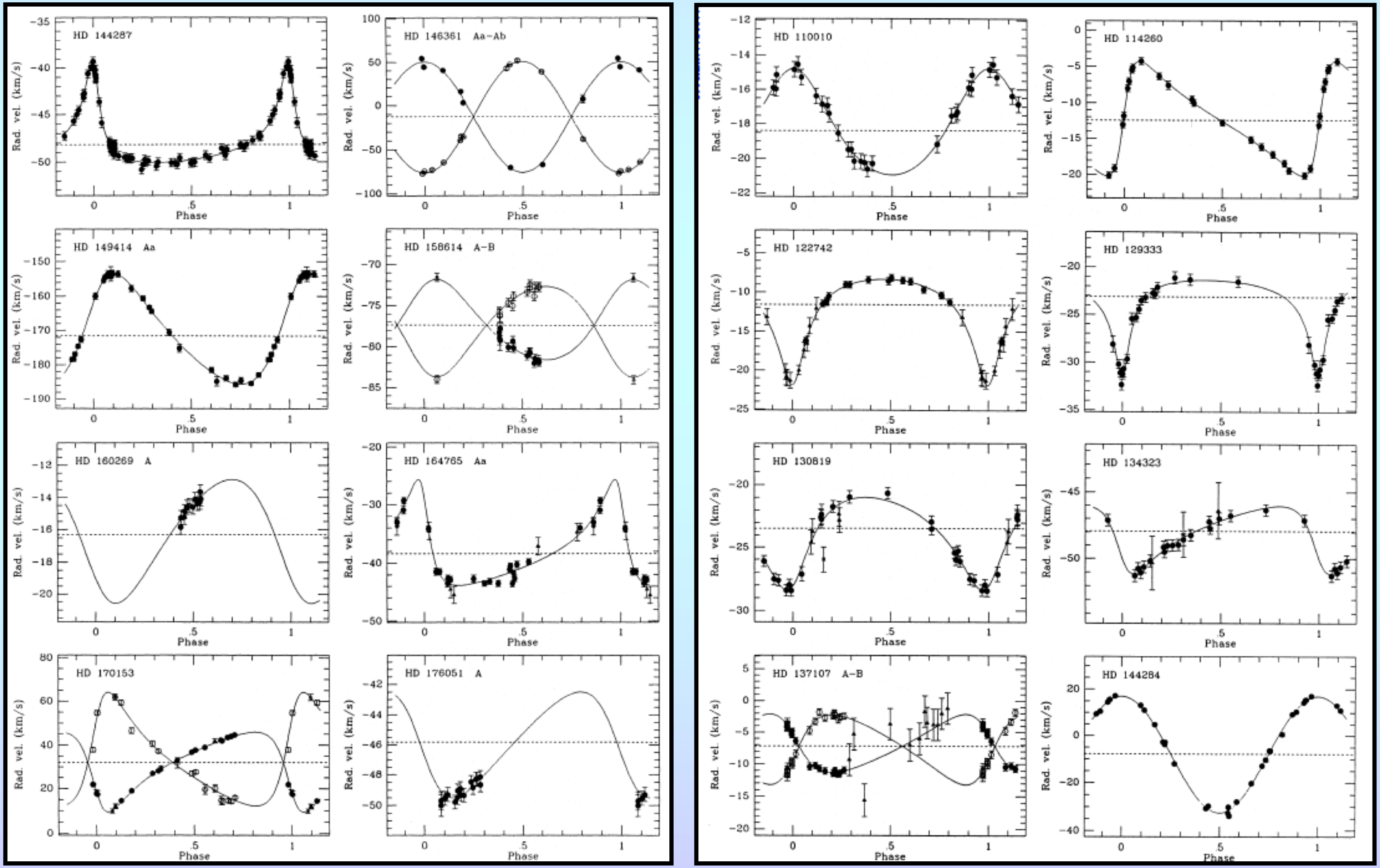
In other words, from a single-line spectroscopy binary, it is impossible to determine individual masses, mass ratios, or even total mass. Note also that the mass function depends on $\sin^3 i$.

$$f\left(M_2\right)=\frac{M_2^3}{\left(M_2+M_1\right)^2}\sin^3 i$$

This (with or without the $\sin^3 i$ term) is called the mass function

# Radial Velocity Curves for Spectroscopic Binaries

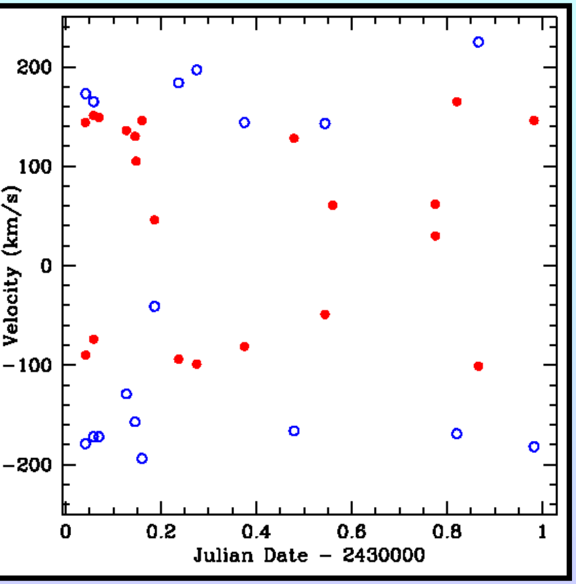
Depending on the eccentricity, the radial velocity curve of a binary can have many shapes; the closer to sinusoidal, the closer $\varepsilon = 0^\circ$ .



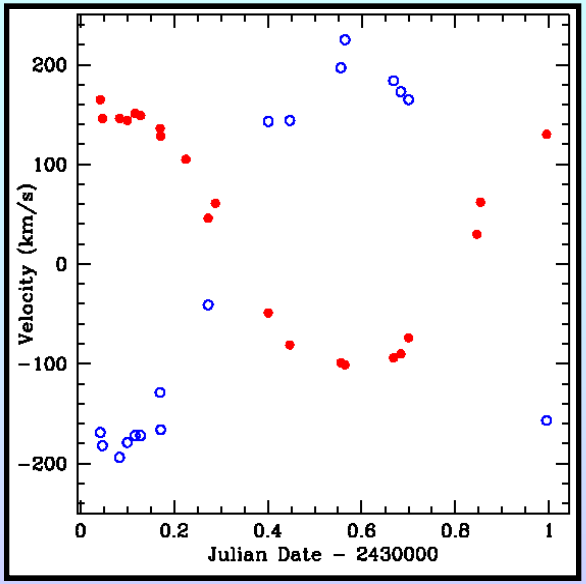
# Note on Radial Velocity Curves

One rarely has single time series containing the entire time period of a binary. Instead, one usually has a series of observations taken at various random phases. One must combine the data to discover the period. (This is best done in Fourier space.) But note: time series are all susceptible to aliasing (i.e., a 12 day period can look like an 18 day period).

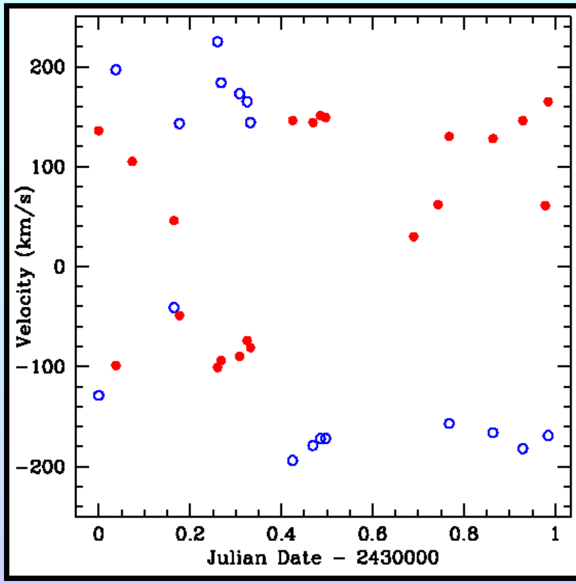
P=1.65 days




P=1.669742 days



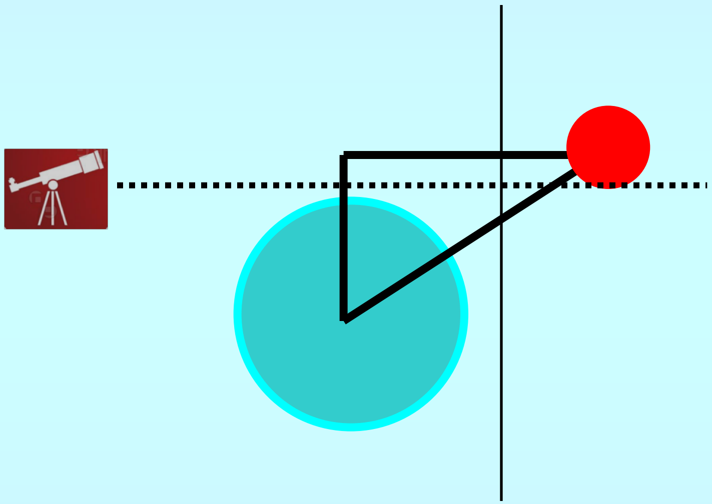

P=1.68 days

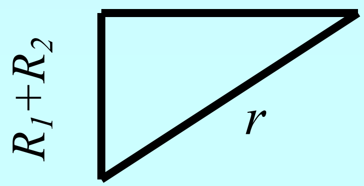


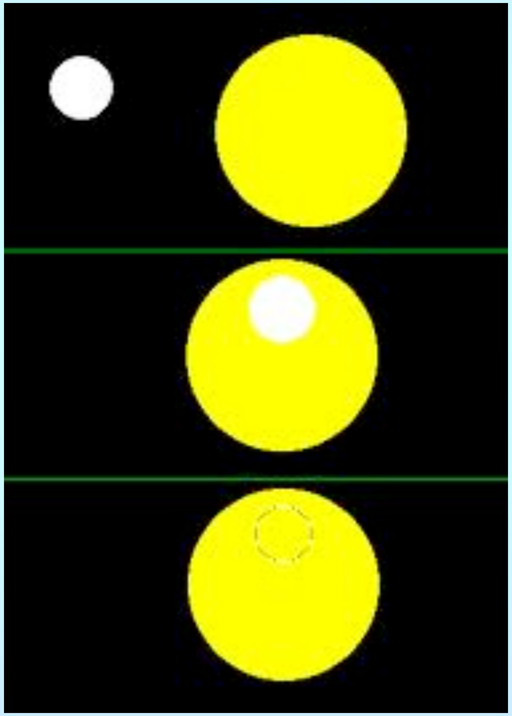
# Eclipsing Binaries

If a binary system is eclipsing, then we have a good estimate of its inclination. If $r$ is the separation of the two stars, then, for at least a partial eclipse

$$|\cos i| < \frac{R_1 + R_2}{r}$$





For a total eclipse to happen $$|\cos i| < \frac{R_1 - R_2}{r}$$

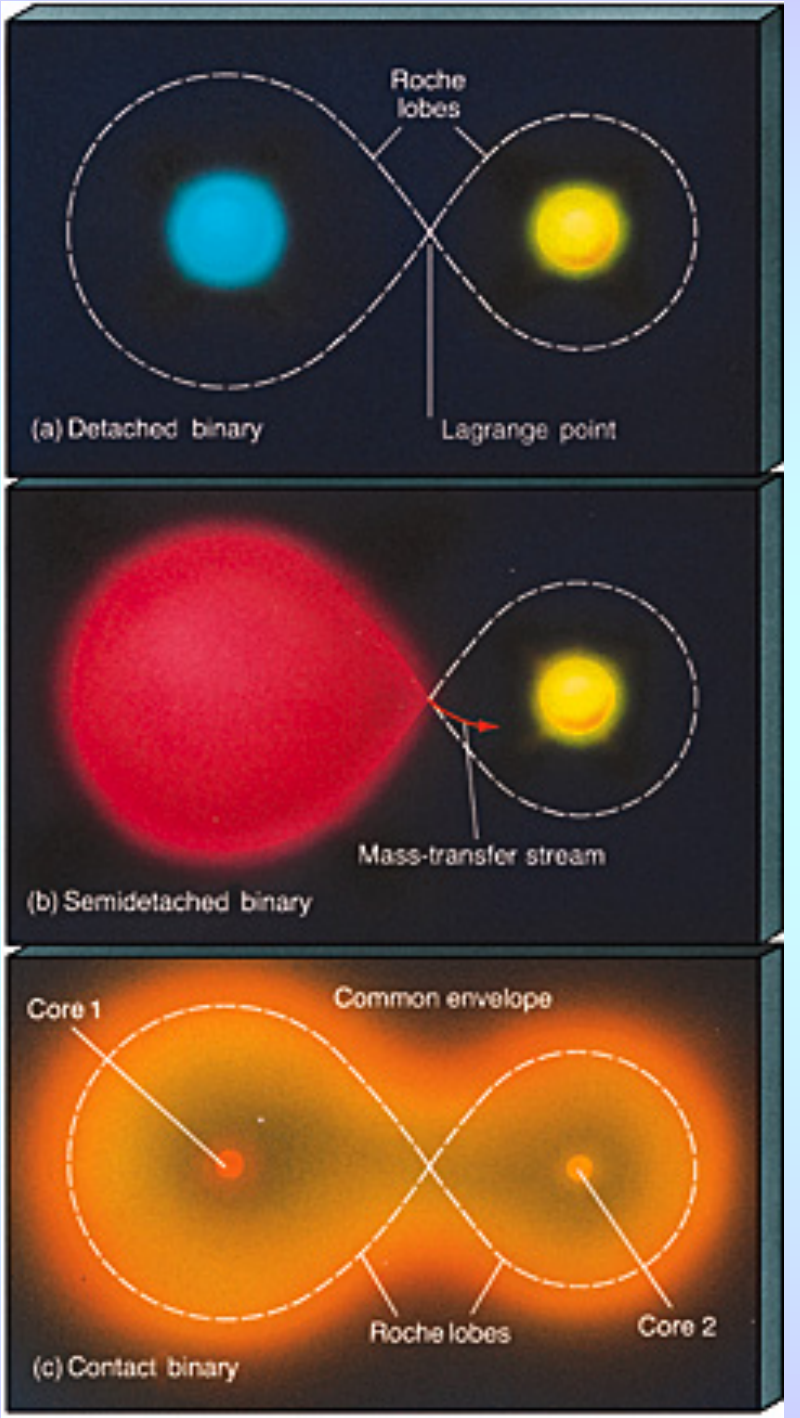
If $r \gg R_1+R_2$ then $i \sim 90^\circ$ . Also, the larger the separation, the rarer the phenomenon. Many eclipsing binaries have short periods.

If the separation becomes too small, the stars can become tidally distorted and/or interact

Detached: the stars are separate and do not affect one another.

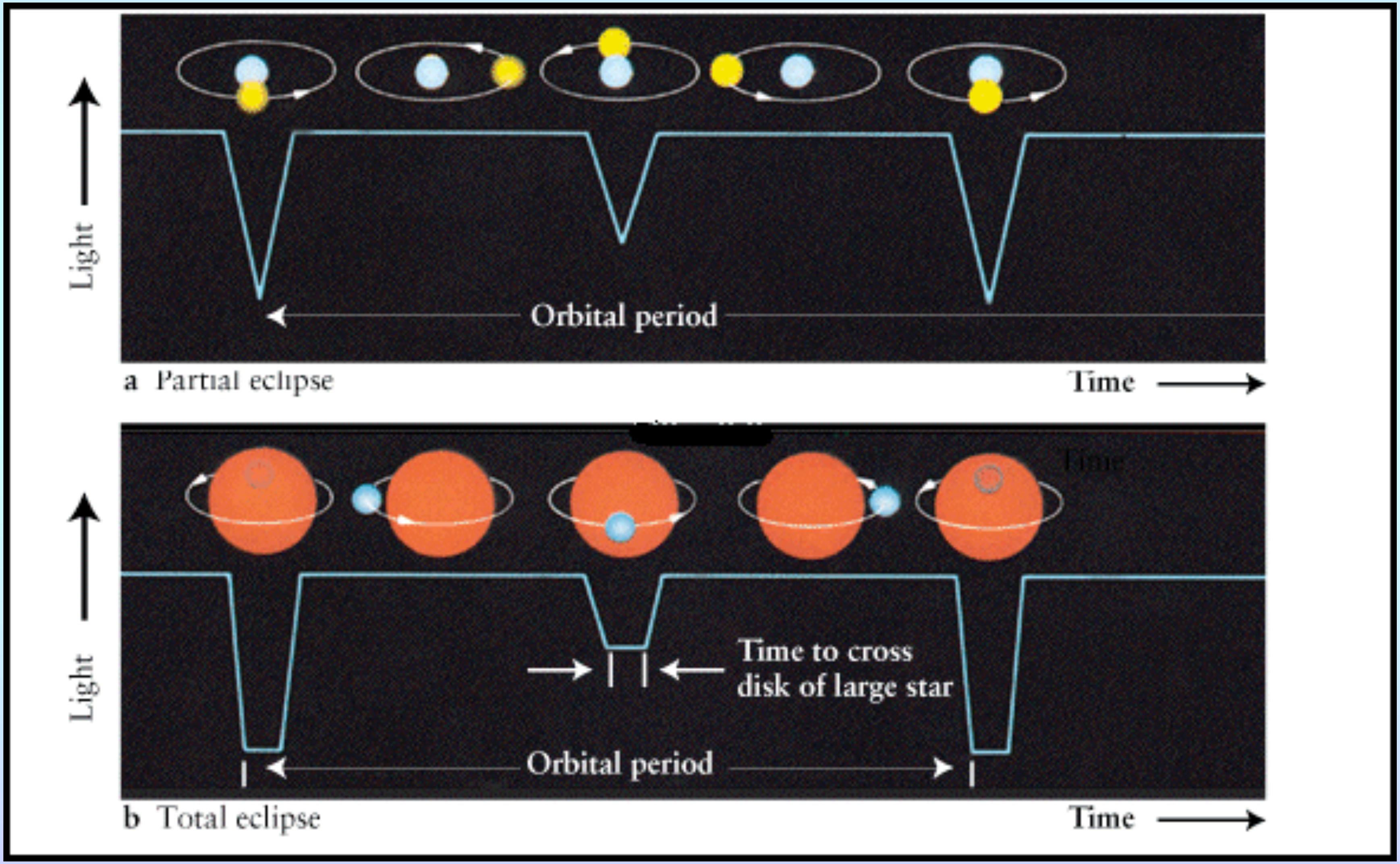
Semi-detached: one star is spilling mass (*i.e.*, accreting) onto the other

Contact: two stars are present inside a common envelope (*i.e.*, it is a common-envelope binary).



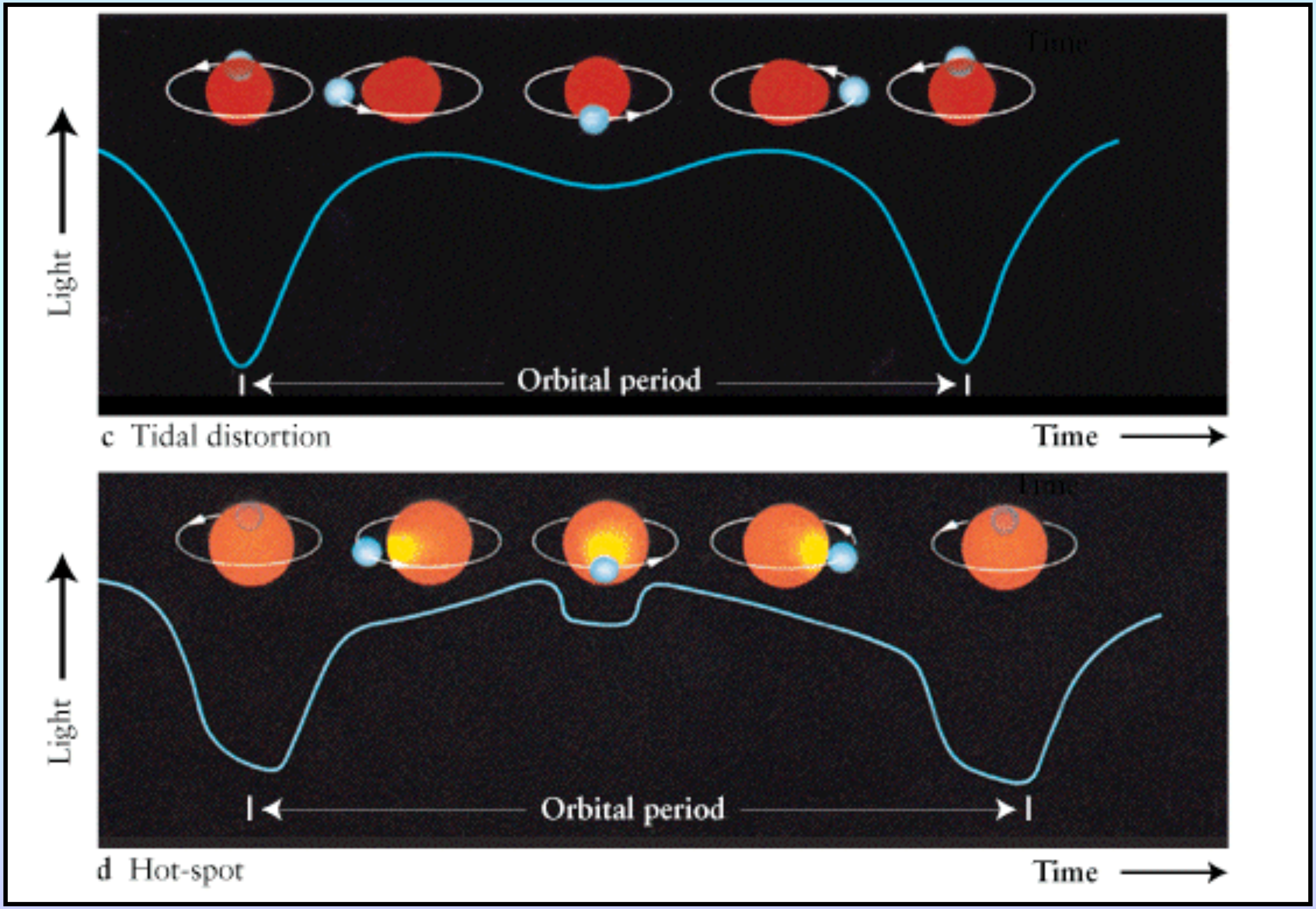
# Partial and Total Eclipses

The shape of the light curve during eclipse defines whether the eclipse is total or partial.



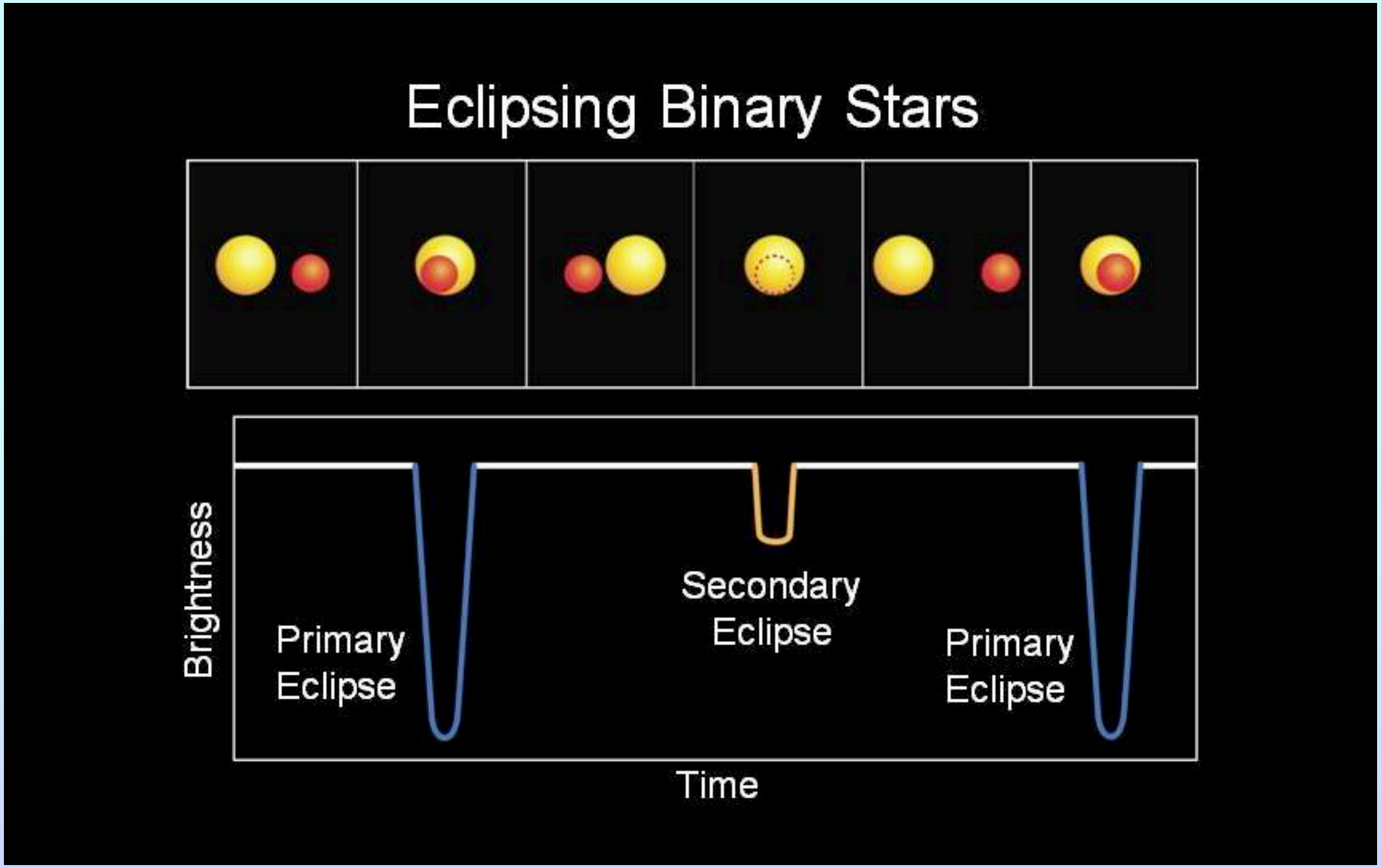
# Partial and Total Eclipses

In practice, other effects, such as star-spots, tidal distortions, limb-darkening, and hot spots may also effect the shape of the light curve.



# Depth of Eclipse

From symmetry the area blocked during the primary eclipse is exactly the same as that of the second eclipse. Consequently, when calculating the amount of light that is eclipsed, $L = 4\pi R^2 \sigma T^4$, the radius doesn't matter – only the temperature counts.



# Total Eclipses

Let $F = 1$ be the relative flux from a system out of eclipse, $F_T$, the relative flux during total eclipse, and $F_a$ the flux during the annular eclipse. If we designate Star 1 to be the larger star, and Star 2 the smaller star, then

During the total eclipse

- $F_1 = F_T$ and $F_2 = F - F_T$

During the annular eclipse

- $F_a = F_1 + F_2 - \left(\frac{\pi R_2}{\pi R_1}\right)^2 F_1 = 1 - \left(\frac{R_2}{R_1}\right)^2 F_1$

Therefore, the ratio of the stellar radii is simply

- $\kappa = \frac{R_2}{R_1} = \left(\frac{1 - F_a}{F_T}\right)^{1/2}$

# Total Eclipses

For systems, with $i = 90^\circ$ , the eclipse begins when

$$\sin\theta_e = \frac{R_1 + R_2}{a}$$

and becomes full when

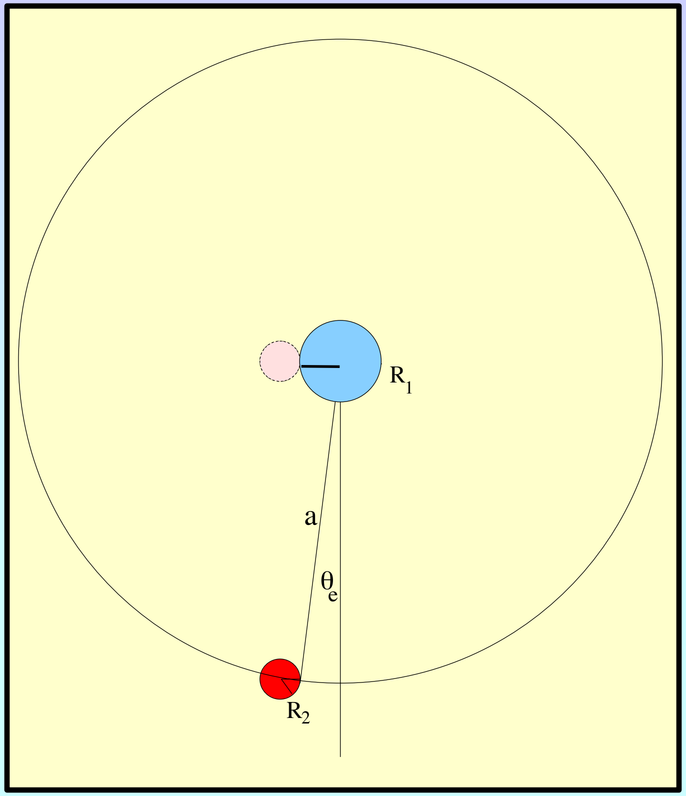
$$\sin\theta_i = \frac{R_1 - R_2}{a}$$

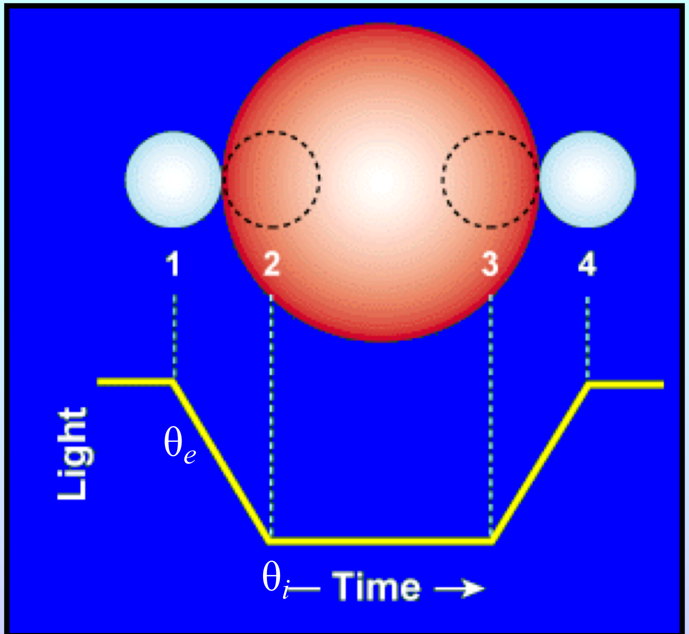
In the more general case ($i \neq 90^\circ$ ), the eclipse beings when the projected separation is

$$\delta_e^2 = \sin^2\theta_e + \cos^2\theta_e \cos^2 i = \left(\frac{R_1 + R_2}{a}\right)$$

and is full when

$$\delta_i^2 = \sin^2\theta_i + \cos^2\theta_i \cos^2 i = \left(\frac{R_1 - R_2}{a}\right)$$



# Solving Eclipsing Binaries

Note that for total eclipses, the relative sizes of the stars and the orbit, along with the inclination of the system, can be computed without any information other than the light curve:

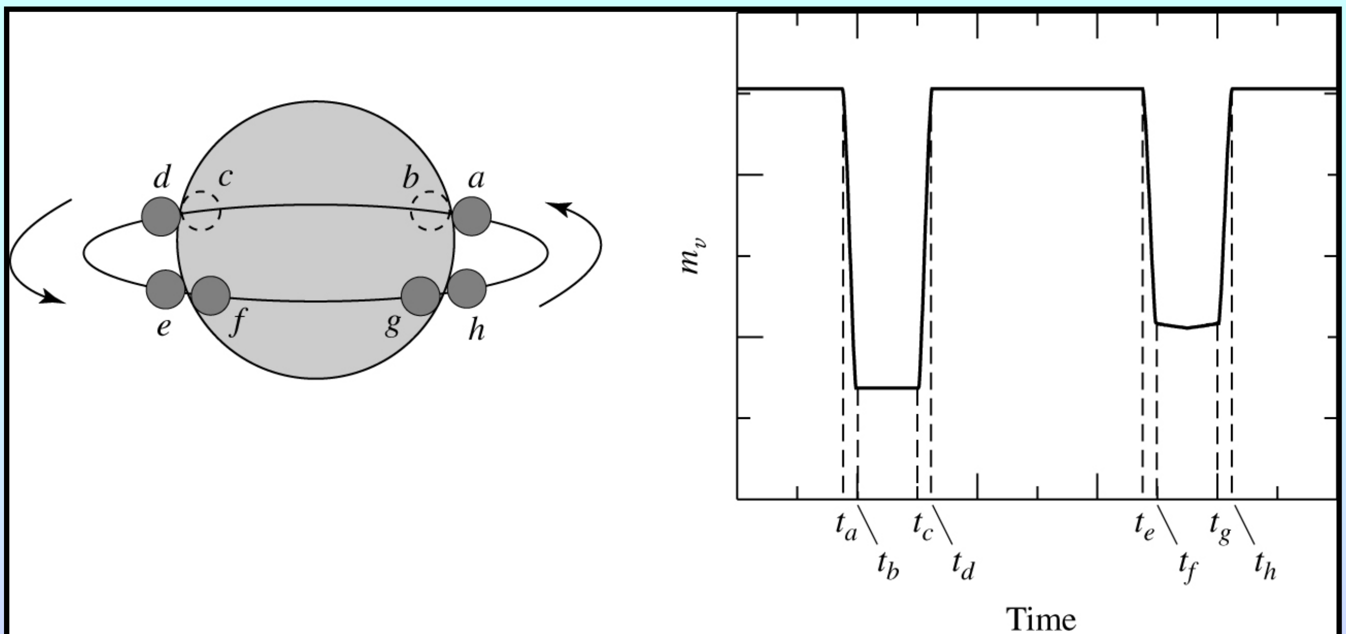
- Depth of total eclipse: $F_1 = 1 - F_2$
- The ratio of the stellar sizes: $\dfrac{R_2}{R_1} = \left(\dfrac{1 - F_a}{F_T}\right)^{1/2}$
- The angle of egress: $\sin^2\theta_e + \cos^2\theta_e \cos^2 i = \left(\dfrac{R_1 + R_2}{a}\right)$
- The angle of ingress: $\sin^2\theta_i + \cos^2\theta_i \cos^2 i = \left(\dfrac{R_1 - R_2}{a}\right)$

If you work with the quantities, $R_2/R_1$, $R_2/a$, $i$, and $F_1/F_2$, there are 4 equations and 4 unknowns. If the system is also a spectroscopic binary, you then you have additional constraints on sin $i$, and can thus measure precise masses and radii.

# Eclipsing Spectroscopic Binaries

If a system is both an eclipsing and spectroscopic binary, then everything can be measured for the system:

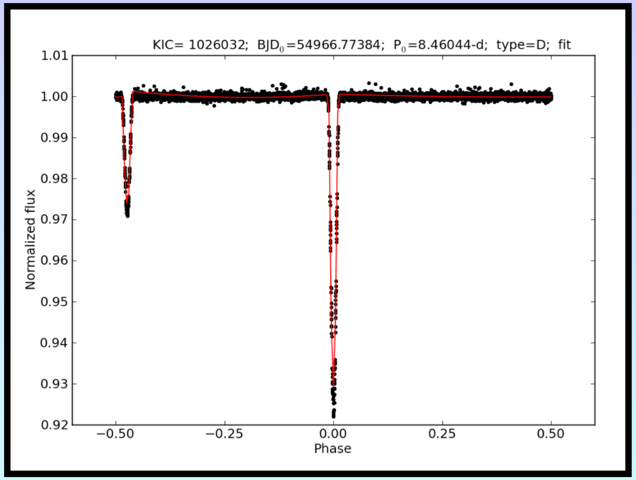
- From the eclipse, you can measure $R_2/R_1$, $R_2/a$, $i$, and $F_1/F_2$
- From the velocities, you can obtain absolute size. For example, if we assume $i \sim 90°$ and a circular orbit, then measuring individual stellar radii is simple.

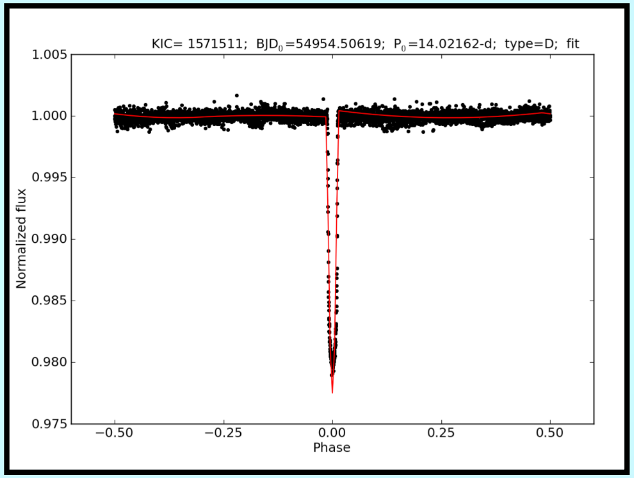


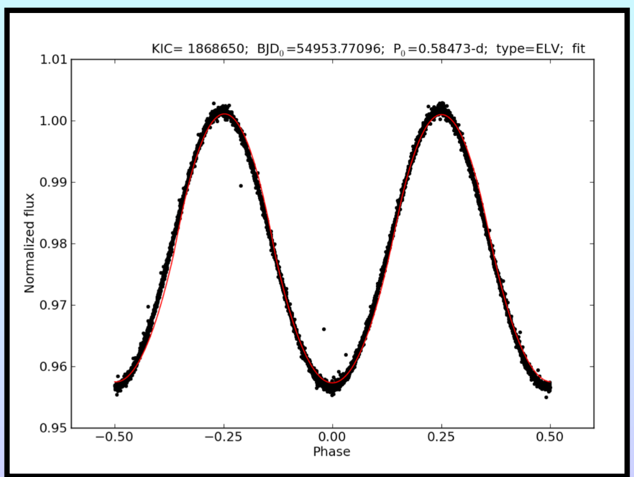

$$R_1 = \frac{v}{2}\left(t_c - t_a\right)$$

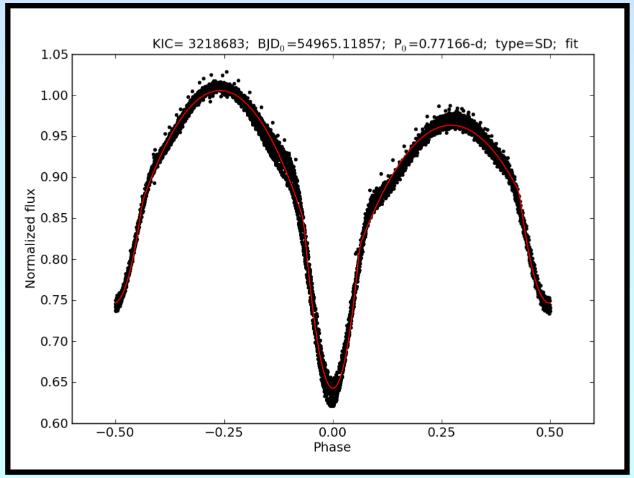
$$R_2 = \frac{v}{2}\left(t_b - t_a\right)$$

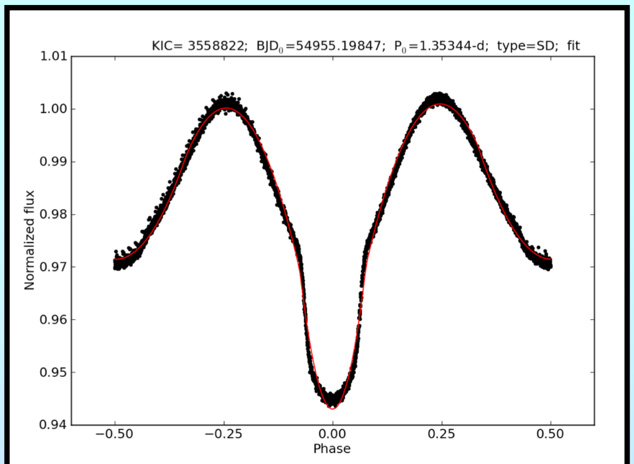
# Kepler Data

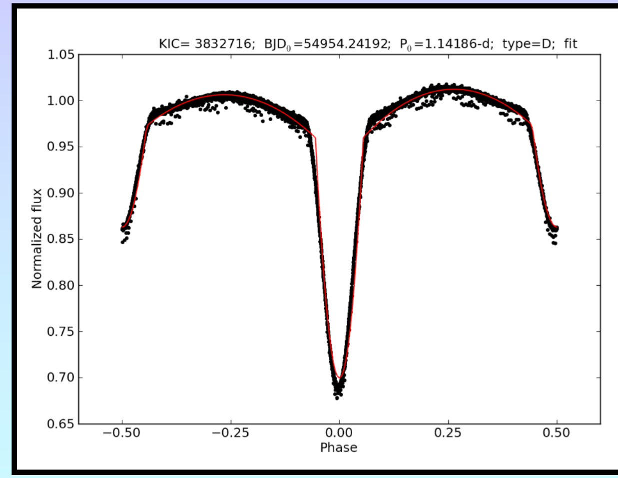

KIC= 1026032; BJD$_0$ =54966.77384; P$_0$=8.46044-d; type=D; fit
Normalized flux
Phase

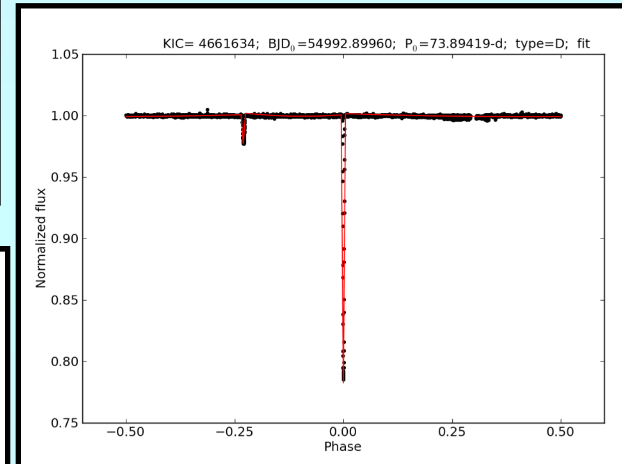

KIC= 1571511; BJD$_0$ =54954.50619; P$_0$=14.02162-d; type=D; fit
Normalized flux
Phase

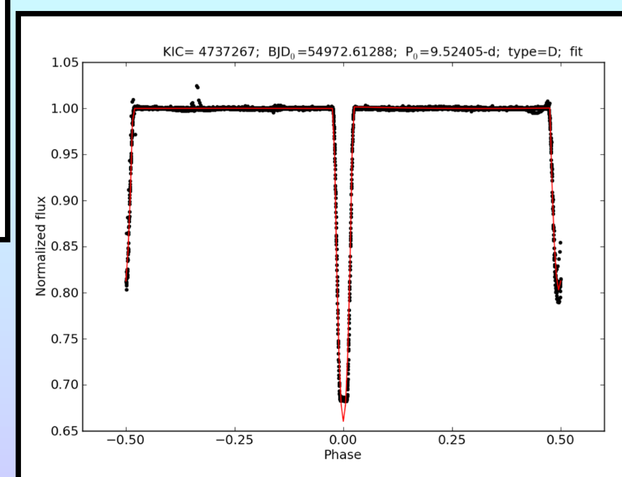

KIC= 1868650; BJD$_0$ =54953.77096; P$_0$=0.58473-d; type=ELV; fit
Normalized flux
Phase


KIC= 3218683; BJD$_0$ =54965.11857; P$_0$=0.77166-d; type=SD; fit
Normalized flux
Phase


KIC= 3558822; BJD$_0$ =54955.19847; P$_0$=1.35344-d; type=SD; fit
Normalized flux
Phase


KIC= 3832716; BJD$_0$ =54954.24192; P$_0$=1.14186-d; type=D; fit
Normalized flux
Phase


KIC= 4661634; BJD$_0$ =54992.89960; P$_0$=73.89419-d; type=D; fit
Normalized flux
Phase


KIC= 4737267; BJD$_0$ =54972.61288; P$_0$=9.52405-d; type=D; fit
Normalized flux
Phase

# Aside – Variable Star Names

The naming convention for variable stars – stars that change their brightness due to eclipses, outbursts, etc., is as byzantine as it gets.

- Stars with existing names from Johann Bayer ($\alpha$ Lyrae or $\delta$ Cepheus) or John Flamsteed (55 Cancri or 51 Pegasi) keep their names.
- The first new variable found in a constellation is given the designation "R" as in R Cor Bor. Then comes "S", "T", etc., through "Z".
- The next (10$^{th}$) new variable discovered in a constellation is given the name "RR", as in RR Lyrae. Then comes "RS", "RT", etc., through "RZ".
- The next (19$^{th}$) new variable discovered in a constellation is given the name "SS", as in SS Cygni. Then comes "ST", "SU", etc., through "SZ". Then "TT", "TU", etc., through "TZ", until finally, "ZZ" as in ZZ Ceti.
- The next (55$^{th}$) new variable discovered starts at the beginning of the alphabet, with the letters, "AA" as in AA Cyg. After that comes "AB" through "AZ", "BB" through "BZ", "CC" through "CZ" until "QQ" through "QZ". (But the letter "J" is always skipped!)
- When the 335$^{th}$ new variable is discovered, the system gives up, and names become V335, V336, etc., as in V1500 Cygni.